Formal Contexts, Formal Concept Analysis, and Galois Connections

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Formal concept analysis (FCA) is built on a special type of Galois connections called polarities. We present new results in formal concept analysis and in Galois connections by presenting new Galois connection results and then applying these to formal concept analysis. We also approach FCA from the perspective of collections of formal contexts. Usually, when doing FCA, a formal context is fixed. We are interested in comparing formal contexts and asking what criteria should be used when determining when one formal context is better than another formal context. Interestingly, we address this issue by studying sets of polarities.

1 Formal Concept Analysis and Order-Reversing Galois Connections

We study formal concept analysis (FCA) from a "larger" perspective than is commonly done. We emphasize formal contexts. For example, we are interested in questions such as if we are working with a given formal context $\mathscr{K}$, that is, we are working with a set of objects G, a set of properties M, and a relation $R \subset G \times M$, what do we do if we want to replace $\mathscr{K}$ with a better formal context. Of course, this raises the question: what makes one formal context better than another formal context. We address this question in multiple ways.

We look at sets of formal contexts such that each formal context has the same set of objects G and the same set of properties M, and we define orderings on these sets of formal contexts. Each of these orderings has a mathematical motivation. We also look at a category in which the objects are formal contexts. This category is rich and well structured. All our results come from using the strong relationship between formal concept analysis and Galois connections. In fact, in getting new ideas and results for FCA and formal contexts, we also get new results for Galois connections.

Formal concept analysis (FCA) consists of methods to analyze data and represent knowledge and is usually done from the perspective of a single formal context. The methods and tools used in FCA are applicable for all formal contexts, but usually one does not look at FCA from the perspective of a collection of formal contexts. In this paper, we look at FCA from the perspective of ordered collections of formal contexts and from the perspective of a category of formal contexts. Our perspective also differs from that of many FCA studies because we think of formal contexts not so much in terms of their usual presentation but in terms of their effective structure. The usual presentation of a formal context is as two

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sets—a set of objects and a set of properties which the objects may have—and a relation which relates or connects an object to the properties which hold for that object. Though this is the usual presentation, what makes FCA useful in data analysis and knowledge representation is the Galois connections each of which is determined by a relation and is defined between the powerset of the set of objects and the powerset of the set of properties. Each of these Galois connections clusters sets of objects in the powerset of objects and clusters sets of attributes in the powerset of attributes, and the Galois connection also naturally "connects" clusters from the two powersets. Thus, we organize our study of FCA in the context or setting of Galois connections.

In this first section, we introduce FCA and Galois connections, and we review properties of each. The review emphasizes that the effective structure of a formal context is the accompanying Galois connection. In Section 2, we define orderings on sets of Galois connections, and these orderings also order our sets of formal contexts. In Section 3, we present our category of formal contexts and show how this category embeds into and, in fact, is structured within a larger category of Galois connections. Section 4 has an FCA example which involves the semantic web and which shows the importance of our goals in studying FCA from a perspective which emphasizes formal contexts. In our conclusion, Section 5, we note that we raise several questions, and we mention possible areas for related future research.

Definition 1.1 A *formal context* is an ordered triple (G,M,R) where G is a set of objects, M is a set of attributes, and R is a relation from G to M, i.e., $R \subset G \times M$.

Definition 1.2 A *Galois connection* is an ordered quadruple $(f,(P,\leq),(Q,\sqsubseteq),g)$ such that $(P,\leq)$ and $(Q,\sqsubseteq)$ are partially ordered sets, and $f : P \to Q$ and $g : Q \to P$ are order-reversing functions such that for each $p \in P$, $p \leq gf(p)$ and for each $q \in Q$, $q \sqsubseteq fg(q)$.

Definition 1.3 (Alternate Definition) A *Galois connection* is an ordered quadruple $(f,(P,\leq),(Q,\sqsubseteq),g)$ such that $(P,\leq)$ and $(Q,\sqsubseteq)$ are partially ordered sets, and for each $p \in P$ and $q \in Q$, $p \leq g(q)$ if and only if $q \sqsubseteq f(p)$.

Galois connections may be defined with order-reversing or order-preserving functions. They were originally defined with order-reversing functions by G. D. Birkhoff [2, 3] in the special case in which the partially ordered sets are powersets with the partial orders being subset inclusion. Birkhoff called these special order-reversing Galois connections *polarities*. Subsequently, O. Ore [13] extended Birkhoff's notion to arbitrary posets and called them *Galois connexions*. It was J. Schmidt [14] who retained the name *Galois connections* but changed the functions to be order-preserving. A Galois connection with order-preserving maps is also called an *adjunction* [10].

In this paper, we use order-reversing maps, which is standard in FCA. Thus, in this paper, a Galois connection is an order-reversing Galois connection.

Notation 1.4 Sometimes for brevity, we may write (f,g) instead of $(f,(P,\leq),(Q,\sqsubseteq),g)$ for a Galois connection.

We use a superscript $\to$ on a function symbol to stand for the corresponding forward powerset function. For example, if $f : X \to Y$, then $f^{\to} : \wp(X) \to \wp(Y)$ such that for $A \subset X$, $f^{\to}(A) = \{f(x) : x \in A\} \subset Y$. This notation was used by T. S. Blyth in [4].

From the above, it is not clear why one would want to study Galois connections and FCA in the same paper, and why one would think that new results in one area should be of interest in the other area. When Birkhoff defined the maps in a polarity, he did it in terms of and based on a relation between the base sets of the two powersets, i.e., he was in an FCA setting; see Theorem 1.8. When Ore extended

Birkhoff's polarities to Galois connections, he also showed that there is a bijection between the set of relations between two sets and the set of polarities or Galois connections between the powersets of the two sets. The bijections are given in Theorem 1.8 and in Construction 2.8.

The following proposition is well known; see, for example, [7] and [12].

Proposition 1.5 Let $(f,(P,\leq),(Q,\sqsubseteq),g)$ be a Galois connection.

1. $fgf = f$ and $gfg = g$.
2. The image points are called fixed points. $p \in g^{\rightarrow}(Q)$ if and only if $p = gf(p)$. Likewise, $q \in f^{\rightarrow}(P)$ if and only if $q = fg(q)$.
3. P and Q are naturally organized or structured by the fibers of f and g, respectively. Each fiber of f contains exactly one point of $g^{\rightarrow}(Q)$, and each fiber of g contains exactly one point of $f^{\rightarrow}(P)$. The image point in each fiber is the largest element of the fiber.
4. The partition of non-empty fibers of P has the same partially ordered structure as $g^{\rightarrow}(Q)$, and the partition of non-empty fibers of Q has the same partially ordered structure as $f^{\rightarrow}(P)$. If E_1 and E_2 are two non-empty fibers or equivalence classes, for example, in P, then $E_1 \leq E_2$ if and only if there exist $p_1 \in E_1$ and $p_2 \in E_2$ such that $p_1 \leq p_2$.
5. $g^{\rightarrow}(Q)$ and $f^{\rightarrow}(P)$ are anti-isomorphic partially ordered sets, and $f|_{g^{\rightarrow}(Q)}^{f^{\rightarrow}(P)} : g^{\rightarrow}(Q) \to f^{\rightarrow}(P)$ and $g|_{f^{\rightarrow}(P)}^{g^{\rightarrow}(Q)} : f^{\rightarrow}(P) \to g^{\rightarrow}(Q)$ are order-reversing bijections. In fact, $f|_{g^{\rightarrow}(Q)}^{f^{\rightarrow}(P)}$ and $g|_{f^{\rightarrow}(P)}^{g^{\rightarrow}(Q)}$ are anti-isomorphic inverses of each other. Hence, the set of fibers in P and the set of fibers in Q are anti-isomorphic partially ordered sets.
6. If P or Q is a [complete] lattice, then so are $g^{\rightarrow}(Q)$ and $f^{\rightarrow}(P)$. However, $g^{\rightarrow}(Q)$ and $f^{\rightarrow}(P)$ need not be sublattices of P and Q, respectively.
7. f and g uniquely determine each other. In fact, for each $p \in P$,

$$f(p) = \bigvee\{q \in Q | p \leq g(q)\},$$

and for each $q \in Q$,

$$g(q) = \bigvee\{p \in P | q \sqsubseteq f(p)\}.$$

Remark 1.6 In Proposition 1.5, P and Q need not be complete lattices or even lattices, and thus, joins and meets in P or Q need not exist. However, the joins in item 7 do exist when (f,g) is a Galois connection.

Terminology 1.7 Based on results in Proposition 1.5, we often use the following terminology: a fiber of f in P or of g in Q is called a *leaf*, elements in the same leaf are called *equivalent*, and the largest element of a leaf is called a *node*. This latter term visually suggests the fact that a leaf attaches to the subset of fixed points in P or Q by its largest element. Also, by item 5 of Proposition 1.5, we say that leaf E in P and leaf F in Q are anti-isomorphic leaves if their nodes are anti-isomorphic nodes.

The following fundamental result from Birkhoff [2, 3] is foundational in FCA, in part because it links the critical information of Proposition 1.5 to the ideas of FCA, as shown throughout this paper.

Theorem 1.8 (Birkhoff Operators). Let G and M be arbitrary sets, and let $R \subset G \times M$ be a relation. Define $H_R : \wp(G) \to \wp(M)$ and $K_R : \wp(M) \to \wp(G)$ by

$$\text{for } S \subset G, H_R(S) = \{m \in M : gRm \forall g \in S\}$$

$$\text{for } T \subset M, K_R(T) = \{g \in G : gRm \forall m \in T\}$$

$(H_R, \wp(G), \wp(M), K_R)$ is a Galois connection where the partial orderings on both $\wp(G)$ and $\wp(M)$ are the subset orderings. When no confusion is likely, we may use H and K in place of H_R and K_R, respectively.

As mentioned above, Birkhoff was the first to define a Galois connection, which he called a polarity. He defined a polarity in terms of the Birkhoff operators, given in Theorem 1.8. Thus, he defined a Galois connection beginning with two sets and a relation between them. It should be noted that for each Galois connection between powersets, there exists a relation between the underlying sets which generates the two order-reversing maps as a pair of Birkhoff operators. Thus, for every pair of sets G and M, there is a bijection between the set of Galois connections between the powersets of G and M and the set of relations from G to M; see Ore [13]. The bijection from relations to Galois connections is given in Theorem 1.8, and its inverse from Galois connections to relations is given in Construction 2.8.

Most of the following FCA definitions and results may be found in [9] or [6].

Definition 1.9 Let $\mathscr{K} := (G,M,R)$ be a formal context. A *formal concept* of the formal context is an ordered pair (A,B) with $A \subset G$ and $B \subset M$ such that $H(A) = B$ and $K(B) = A$. If (A,B) and (A',B') are formal concepts of $\mathscr{K}$, then $(A,B) \leq (A',B')$ if $A \subset A'$ or, equivalently, if $B' \subset B$.

Definition 1.10 Let $\mathscr{K} := (G,M,R)$ be a formal context. The set of all formal concepts of $\mathscr{K}$ with the partial ordering defined in Definition 1.9 is called the *concept lattice of* $\mathscr{K}$.

Theorem 1.11 Let $\mathscr{K} := (G,M,R)$ be a formal context, and let $(H, \wp(G), \wp(M), K)$ be the associated Galois connection. The concept lattice of $\mathscr{K}$ is a complete lattice; it is isomorphic to $K^{\rightarrow}(\wp(M))$ and anti-isomorphic to $H^{\rightarrow}(\wp(G))$.

The following definitions of formal preconcept and formal protoconcept come from [17].

Definition 1.12 Let $\mathscr{K} := (G,M,R)$ be a formal context.

1. A *formal preconcept* of the formal context is an ordered pair (C,D) with $C \subset G$ and $D \subset M$ such that $C \subset K(D)$ or, equivalently, $D \subset H(C)$.
2. If (C,D) and (C',D') are formal preconcepts of a formal context, then $(C,D) \sqsubseteq (C',D')$ if $C \subset C'$ and $D \subset D'$. (This partial order on formal preconcepts is not an extension of the partial order on formal concepts.)
3. Let (C,D) be a formal preconcept of $\mathscr{K}$. The collection of all formal concepts (A,B) such that $(C,D) \sqsubseteq (A,B)$ is a subset of the concept lattice of $\mathscr{K}$ and is denoted by $Precon(C,D)$. We order the elements of $Precon(C,D)$ by the partial ordering on formal concepts, i.e., by $\leq$.

Formal preconcepts may be thought of as specifying formal concepts in the sense that formal preconcept (C,D) specifies or "determines" formal concept (A,B) if $(C,D) \sqsubseteq (A,B)$. However, as $Precon(C,D)$ is a subset of formal concepts, we use the formal concept partial ordering on $Precon(C,D)$. Also, note that usually a preconcept does not uniquely specify or determine a formal concept, i.e., $|Precon(C,D)|$ is usually greater than one, and $Precon(C,D)$ is never empty. $Precon(C,D)$ always contains at least $(KH(C),H(C))$ and $(K(D),HK(D))$ though these two formal concepts may be equal.

Proposition 1.13 Let $\mathscr{K} := (G,M,R)$ be a formal context.

1. If (C,D) is a formal preconcept, then $Precon(C,D)$ is itself a complete lattice with $(KH(C),H(C))$ being the smallest formal concept in $Precon(C,D)$ and $(K(D),HK(D))$ being the largest.

2. For formal preconcept (C,D), every formal concept (A,B) with $KH(C) \subset A \subset K(D)$ is in $Precon(C,D)$.
3. For a formal preconcept (C,D), $|Precon(C,D)| = 1$ if and only if $(KH(C),H(C)) = (K(D),HK(D))$ if and only if $KH(C) = K(D)$ if and only if $H(C) = HK(D))$.
4. For a formal preconcept (C,D), $|Precon(C,D)| = 1$ if and only if C and D are in anti-isomorphic leaves.
5. If (C,D) be a formal preconcept, then (C,D) is less than or equal to exactly one formal concept (A,B) in the preconcept partial ordering (i.e., there is a unique formal concept (A,B) with $(C,D) \sqsubseteq (A,B)$) if and only if $(A,B) = (K(D), H(C))$.

Thinking of formal preconcepts as specifying formal concepts, leads to the next proposition.

Proposition 1.14 Let $\mathscr{K} := (G,M,R)$ be a formal context, and let (C,D) and (C',D') be formal preconcepts with $(C,D) \sqsubseteq (C',D')$. Then $Precon(C',D') \subset Precon(C,D)$. Formal preconcept (C',D') is maximal in the $\sqsubseteq$ partial order if and only if $|Precon(C',D')| = 1$.

The higher a formal preconcept is in the preconcept partial ordering, the more specific or precise it is in specifying formal concepts. However, it is not the case that $(C,D) \sqsubseteq (C',D')$ if and only if $Precon(C',D') \subset Precon(C,D)$. Moreover, it may be the case that (C,D) and (C',D') are both maximal and $Precon(C,D) = Precon(C',D')$, but (C,D) and (C',D') are not comparable. For example, if $D = D'$ and if C and C' are not comparable in the subset ordering but are in the same leaf, then $Precon(C,D) = Precon(C',D')$ but (C,D) and (C',D') are not comparable in the formal preconcept partial order. Moreover, if the leaf of C and the leaf of D and D' are equivalent, then (C,D) and (C',D') are both maximal but still not comparable in the preconcept partial order.

Since the formal preconcepts may be thought of as specifications for the formal concepts, we propose a pre-order which is defined on the set of formal preconcepts of a formal context and which more precisely reflects the thinking that formal preconcepts are specifications for formal concepts.

Definition 1.15 Let (C,D) and (C',D') be formal preconcepts of a formal context $\mathscr{K} := (G,M,R)$. Then $(C,D) \preceq (C',D')$ if $Precon(C',D') \subset Precon(C,D)$.

Proposition 1.16 Let (C,D) and (C',D') be formal preconcepts of a formal context $\mathscr{K} := (G,M,R)$. Then $(C,D) \preceq (C',D')$ if and only if $K(D') \subset K(D)$ and $H(C') \subset H(C)$.

Proposition 1.17 Let $\mathscr{K} := (G,M,R)$ be a formal context, and let (C,D) and (C',D') be formal preconcepts. The following are equivalent:

- (C,D) and (C',D') are equivalent as formal preconcepts in the pre-order $\preceq$;
- $(C,D) \preceq (C',D')$ and $(C',D') \preceq (C,D)$;
- $Precon(C,D) = Precon(C',D')$;
- C and C' are in the same equivalence class of $\wp(G)$ and D and D' are in the same equivalence class of $\wp(M)$.

Question 1.18 What is the partial order gotten from the pre-order in Definition 1.15 by equating ordered pairs which are equivalent in the pre-order?

To help us answer this question, we use the following notation.

Notation 1.19 When $(f,(P,\leq),(Q,\sqsubseteq),g)$ is a Galois connection,, we $\mathscr{L}(P)$ and $\mathscr{L}(Q)$ to denote the leaves of P and Q, respectively. Thus, $\mathscr{L}(\wp(G))$ and $\mathscr{L}(\wp(M))$ denote the sets of leaves of the polarity $(H,\wp(G),\wp(M),K)$. Further, we use $f^*:\mathscr{L}(P)\to\mathscr{L}(Q)$ for the map which maps each element in $\mathscr{L}(P)$ to its anti-isomorphic leaf in $\mathscr{L}(Q)$. Likewise, we have the maps $g^*:\mathscr{L}(Q)\to\mathscr{L}(P)$, $H^*:\mathscr{L}(\wp(G))\to\mathscr{L}(\wp M)$, and $K^*:\mathscr{L}(\wp(M))\to\mathscr{L}(\wp(G))$. From items 4 and 5 of Proposition 1.5, we know that f^* and g^* are anti-isomorphic inverses of each other. Similarly, H^* and K^* are anti-isomorphic inverses. When more than one Galois connection is being discussed, we will use $\mathscr{L}(f,P)$ and $\mathscr{L}(g,Q)$ instead of $\mathscr{L}(P)$ and $\mathscr{L}(Q)$, respectively.

Answer 1.20 The resulting partial order is isomorphic to a partial order on a subset of $\wp(G)\times\wp(M)$. If E is a leaf in $\wp(G)$ and F is a leaf in $\wp(M)$, we want (E,F) to be in this subset of $\wp(G)\times\wp(M)$ if and only if $E\leq H^*(F)$ and $F\leq K^*(E)$, where this $\leq$ is defined in item 4 of Proposition 1.5. We let $\mathscr{G}\mathscr{M}$ denote this subset of $\wp(G)\times\wp(M)$.

Saying $(E,F)\in\mathscr{G}\mathscr{M}$ is equivalent to saying there exists C in E and there exists D in F with (C,D) a formal preconcept in $\mathscr{K}=(G,M,R)$, and this is equivalent to saying for every C in E and every D in F, (C,D) is a formal preconcept in $\mathscr{K}$.

For $(E,F),(E',F')\in\mathscr{G}\mathscr{M}$, $(E,F)\leq(E',F')$ if $K^*(F')\leq K^*(F)$ and $H^*(E')\leq H^*(E)$. Compare with Proposition 1.16.

Definition 1.21 Let (G,M,R) be a formal context. A *formal protoconcept* of the formal context is a formal preconcept (C,D) such that *Precon*(C,D) contains exactly one formal concept.

Theorem 1.22 Let (G,M,R) be a formal context; let (H,K) be the associated Galois connection of Birkhoff operators; and let $C\subset G$, $D\subset M$. The following are equivalent:

1. (C,D) is a formal protoconcept.
2. *Precon*(C,D) contains exactly one formal concept.
3. C and D are members of anti-isomorphic leaves.
4. $(K(D),H(C))$ is a formal concept of (G,M,R).
5. $KH(C)=K(D)$.
6. $HK(D)=H(C)$.

Statement (4) of Theorem 1.22 justifies the term "protoconcept"; and in practice, (5) and (6) seem the most convenient to apply.

2 Orderings on Sets of Formal Contexts with the Same Underlying Sets

We are interested in sets of formal contexts which have the the same set of objects and the same set of properties, and in particular, we are interested in orderings on these sets of formal contexts. Studying sets of formal contexts with the same sets of objects and properties is equivalent to studying sets of polarities in which all polarities have the same "first" powerset and all have the same "second" powerset. Further, if we think in terms of Galois connections instead of just polarities, then we can study sets of Galois connections for which the "first" partially ordered set is, for example, $(P,\leq)$ and the "second" partially ordered set is, for example, $(Q,\sqsubseteq)$.

2.1 A Partial Ordering on $\mathscr{G}$

Notation 2.1 Let $(P,\leq)$ and $(Q,\sqsubseteq)$ be partially ordered sets. We let $\mathscr{G}(P,Q)$ or simply $\mathscr{G}$ denote the set of all Galois connections between $(P,\leq)$ and $(Q,\sqsubseteq)$.

For our first ordering on $\mathscr{G}$, we use the partial order on Q and extend it pointwise to the first maps in the elements of $\mathscr{G}$.

Definition 2.2 Let $(f_1,g_1),(f_2,g_2)\in\mathscr{G}$. $(f_1,g_1)\leq(f_2,g_2)$ if for each $p\in P$, $f_1(p)\sqsubseteq f_2(p)$.

Proposition 2.3 Let $(f_1,g_1),(f_2,g_2)\in\mathscr{G}$. $(f_1,g_1)\leq(f_2,g_2)$ if and only if for each $q\in Q$, $g_1(q)\leq g_2(q)$.

Proof: Suppose $(f_1,g_1)\leq(f_2,g_2)$. Let $q\in Q$. $g_1(q)=\bigvee\{p\in P|q\sqsubseteq f_1(p)\}$ and $g_2(q)=\bigvee\{p\in P|q\sqsubseteq f_2(p)\}$. Since for each $p\in P$, $f_1(p)\sqsubseteq f_2(p)$, then $\{p\in P|q\sqsubseteq f_1(p)\}\subset\{p\in P|q\sqsubseteq f_2(p)\}$. Hence, for each $q\in Q$, $\bigvee\{p\in P|q\sqsubseteq f_1(p)\}\leq\bigvee\{p\in P|q\sqsubseteq f_2(p)\}$, and $g_1(q)\leq g_2(q)$.

We proved if for each $p\in P$, $f_1(p)\sqsubseteq f_2(p)$, then for each $q\in Q$, $g_1(q)\leq g_2(q)$. Since the definition of Galois connections is symmetric with respect to f and g, then if we have for each $q\in Q$ that $g_1(q)\leq g_2(q)$, it will follow for each $p\in P$ that $f_1(p)\sqsubseteq f_2(p)$, i.e., it follows that $(f_1,g_1)\leq(f_2,g_2)$. •

Notation 2.4 We use $\mathscr{P}(G,M)$ to denote $\mathscr{G}(\wp(G),\wp(M))$.

The ordering $\leq$ on $\mathscr{G}$ is natural, and since $\sqsubseteq$ is a partial order on Q, then $(\mathscr{G},\leq)$ is also a partially ordered set. Interestingly, there is an equally natural way to define $\leq$ on $\mathscr{P}(G,M)$. This new definition, see below, emphasizes the standard presentation of formal contexts.

As mentioned above, there is a bijection from the set of all relations from G to M to the set of all polarities "built from" G and M, i.e., to the set of all Galois connections from $\wp(G)$ to $\wp(M)$.

Let $\mathscr{R}(G,M)$ be the set of all relations from G to M. Define $\mathscr{F}:\mathscr{R}(G,M)\to\mathscr{P}(G,M)$ by $\mathscr{F}(R)=(H_R,\wp(G),\wp(M),K_R)$ as defined in Theorem 1.8.

Definition 2.5 Define the ordering $\leq_{\mathscr{P}}$ on $\mathscr{P}(G,M)$ such that if $(H_1,K_1),(H_2,K_2)\in\mathscr{P}(G,M)$, then $(H_1,K_1)\leq_{\mathscr{P}}(H_2,K_2)$ if and only if $\mathscr{F}^{-1}(H_1,K_1)\subset\mathscr{F}^{-1}(H_2,K_2)$. Thus, if R_1 is the relation from G to M such that $\mathscr{F}(R_1)=(H_1,K_1)$ and R_2 is the relation G to M such that $\mathscr{F}(R_2)=(H_2,K_2)$ and $R_1\subset R_2$, then $(H_1,K_1)\leq_{\mathscr{P}}(H_2,K_2)$.

Proposition 2.6 The partial order $\leq$ on $\mathscr{G}(\wp(G),\wp(M))$ and the order $\leq_{\mathscr{P}}$ on $\mathscr{P}(G,M)$ are the same.

Proof: Suppose that $(H_1,K_1),(H_2,K_2)\in\mathscr{P}(G,M)$ with $R_1=\mathscr{F}^{-1}(H_1,K_1)$, and $R_2=\mathscr{F}^{-1}(H_2,K_2)$. Further suppose that $(H_1,K_1)\leq_{\mathscr{P}}(H_2,K_2)$, and let $S\in\wp(G)$.

$$H_1(S)=\{m\in M:gR_1m\forall g\in S\}$$

and

$$H_2(S)=\{m\in M:gR_2m\forall g\in S\}.$$

Since $R_1\subset R_2$, then $\{m\in M:gR_1m\forall g\in S\}\subset\{m\in M:gR_2m\forall g\in S\}$. Thus, $H_1\leq H_2$, and $(H_1,K_1)\leq(H_2,K_2)$.

If we begin with $(H_1,K_1)\leq(H_2,K_2)$, then for each $S\in\wp(G)$, we will have $H_1(S)\subset H_2(S)$. Thus, in particular, for each $g\in G$, we have $H_1(\{g\})\subset H_2(\{g\})$, and this implies that $\mathscr{F}^{-1}(H_1,K_1)\subset\mathscr{F}^{-1}(H_2,K_2)$. (See Construction 2.8.) •

Since $\leq$ and $\leq_{\mathscr{P}}$ are the same on $\mathscr{P}(G,M)$, we will use $\leq$ for both.

Proposition 2.7 Let $(f,(P,\leq),(Q,\sqsubseteq),g) \in \mathscr{G}$. If $(P,\leq)$ and $(Q,\sqsubseteq)$ both have greatest elements, $\top_P$ and $\top_Q$, respectively, then $(\mathscr{G},\leq)$ has a greatest element. If $(P,\leq)$ and $(Q,\sqsubseteq)$ both also have least elements, $\bot_P$ and $\bot_Q$, respectively, then $(\mathscr{G},\leq)$ also has a least element.

Proof: If $(P,\leq)$ and $(Q,\sqsubseteq)$ both have greatest elements, then $(f_\top, g_\top)$ defined by

$$\forall p \in P,\ f_\top(p) = \top_Q$$

and

$$\forall q \in Q,\ g_\top(q) = \top_P$$

is the greatest element in $\mathscr{G}$.

If $(P,\leq)$ and $(Q,\sqsubseteq)$ both also have least elements, then $(f_\bot, g_\bot)$ defined by

$$f_\bot(p) = \begin{cases} \bot_Q & \text{if } p \neq \bot_P \\ \top_Q & \text{if } p = \bot_P \end{cases}$$

and

$$g_\bot(q) = \begin{cases} \bot_P & \text{if } q \neq \bot_Q \\ \top_P & \text{if } q = \bot_Q \end{cases}$$

is the least element in $\mathscr{G}$. •

Construction 2.8 Before we define another ordering on $\mathscr{G}$, we want to give the explicit definition of $\mathscr{F}^{-1} : \mathscr{P}(G,M) \to \mathscr{R}(G,M)$. Let $(H,\wp(G),\wp(M),K) \in \mathscr{P}(G,M)$. $\mathscr{F}^{-1}(H,K) = (G,M,R)$ where $(g,m) \in R$ if and only if $m \in H(\{g\})$ if and only if $g \in K(\{m\})$.

2.2 Other Orderings on $\mathscr{G}$

If $(f,(P,\leq),(Q,\sqsubseteq),g)$ is a Galois connection, then for each $p \in P$, $p \leq gf(p)$, and for each $q \in Q$, $q \sqsubseteq fg(q)$. These inequalities can be given by $1_P \leq gf$ and $1_Q \sqsubseteq fg$. These inequalities lead one to think of trying to minimize the differences between 1_P and gf and between 1_Q and fg, which in turn leads, among other things, to the following ordering on $\mathscr{G}$.[1]

Definition 2.9 Let $(f_1,g_1),(f_2,g_2) \in \mathscr{G}(P,Q)$. Define $(f_1,g_1) \preceq (f_2,g_2)$ if and only if for each $p \in P$, $g_2 f_2(p) \leq g_1 f_1(p)$.

Notation 2.10 Let $(f,(P,\leq),(Q,\sqsubseteq),g)$ be a Galois connection. We use $\mathscr{N}(f,P)$ and $\mathscr{N}(Q,g)$ to denote the sets of nodes of P and Q, i.e., the sets of image points of g and f, respectively.

Proposition 2.11 Let $(f_1,g_1),(f_2,g_2) \in \mathscr{G}(P,Q)$. The following are equivalent.

1. $(f_1,g_1) \preceq (f_2,g_2)$
2. $\mathscr{N}(f_1,P) \subset \mathscr{N}(f_2,P)$
3. The partition of P by f_2 is a refinement of the partition of P by f_1, i.e., for each $E_2 \in \mathscr{L}(f_2,P)$, there is an $E_1 \in \mathscr{L}(f_1,P)$ such that $E_2 \subset E_1$.

[1]This ordering was suggested by C. Rohwer during a conversation at the University of Stellenbosch, South Africa.

Proof: (1. $\Rightarrow$ 2.) Let $p^* \in \mathscr{N}(f_1, P)$. Then $p^* = g_1 f_1(p^*)$. Since $p^* \leq g_2 f_2(p^*) \leq g_1 f_1(p^*)$, then $p^* = g_2 f_2(p^*)$, and thus, $p^* \in \mathscr{N}(f_2, P)$.

(2. $\Rightarrow$ 1.) Let $p \in P$. Then $p \leq g_1 f_1(p)$. Since both f_2 and g_2 are order-reversing, then $g_2 f_2$ is order-preserving. Therefore, $g_2 f_2(p) \leq g_2 f_2 g_1 f_1(p)$. However, $g_1 f_1(p) \in \mathscr{N}(f_1, P) \subset \mathscr{N}(f_2, P)$. It follows that $g_2 f_2 g_1 f_1(p) = g_1 f_1(p)$, and therefore, $g_2 f_2(p) \leq g_1 f_1(p)$.

(1. $\Rightarrow$ 3.) To show that $\mathscr{L}(f_2, P)$ is a refinement of $\mathscr{L}(f_1, P)$, we begin with $p_1, p_2 \in P$ with $f_2(p_1) = f_2(p_2)$, and we show that $f_1(p_1) = f_1(p_2)$. Since $f_2(p_1) = f_2(p_2)$, then $g_2 f_2(p_1) = g_2 f_2(p_2)$. It follows that $p_1 \leq g_2 f_2(p_1) \leq g_1 f_1(p_1)$, and therefore, $f_1 g_1 f_1(p_1) \sqsubseteq f_1 g_2 f_2(p_1) \sqsubseteq f_1(p_1)$. However, since $f_1(p_1) = f_1 g_1 f_1(p_1)$, then $f_1(p_1) = f_1 g_2 f_2(p_1)$. Likewise, $f_1(p_2) = f_1 g_2 f_1(p_2)$. Since $g_2 f_2(p_1) = g_2 f_2(p_2)$, then $f_1 g_2 f_2(p_1) = f_1 g_2 f_2(p_2)$, and $f_1(p_1) = f_1(p_2)$. $\bullet$

(3. $\Rightarrow$ 1.) Let $p \in P$. Assume $p \in E_2 \in \mathscr{L}(f_2, P)$, and assume $p \in E_1 \in \mathscr{L}(f_1, P)$. Since p is in both E_2 and E_1 and since $\mathscr{L}(f_2, P)$ refines $\mathscr{L}(f_1, P)$, then $E_2 \subset E_1$. Since $g_2 f_2(p)$ is the largest element in E_2 and $g_1 f_1(p)$ is the largest element in E_1, then $g_2 f_2(p) \leq g_1 f_1(p)$. $\bullet$

If $(P, \leq)$ and $(Q, \sqsubseteq)$ have greatest elements, $\top_P$ and $\top_Q$, respectively, then the least element in $(\mathscr{G}, \preceq)$ is (f, g) such that both f and g map everything to $\top_Q$ and $\top_P$, respectively. Interestingly, this (f, g) is the greatest element in $(\mathscr{G}, \leq)$.

(f', g') is a maximal element in $(\mathscr{G}, \preceq)$ if $1_P = g' f'$. $(\mathscr{G}, \preceq)$ may have multiple maximal elements, and if (f', g') and (f'', g'') are both maximal elements, then $(f', g') \preceq (f'', g'')$ and $(f'', g'') \preceq (f', g')$. Thus, $\preceq$ is only a pre-ordering on $\mathscr{G}$. Example 2.15 shows that maximal elements are not unique.

Given the symmetric nature of Galois connections, one might think that if $(f_2, g_2$ produces a finer partition on P than does (f_1, g_1), then (f_2, g_2) would also produce a finer partition on Q. In other words, one might think that the statement for every $p \in P$, $g_2 f_2(p) \leq g_1 f_1(p)$ is equivalent to the state for every $q \in Q$, $f_2 g_2(q) \sqsubseteq f_1 g_1(q)$. However, as the next example, Example 2.12, shows this is not the case.

Example 2.12 Let $P = \{1, 2, 3\}$, and let $Q = \{1, 2, 3, 4\}$ with the usual ordering on natural numbers. For f_1, define $f_1(3) = f_1(2) = 2$ and $f_1(1) = 4$. Thus, g_1 must be $g_1(4) = g_1(3) = 1$, and $g_1(2) = g_1(1) = 3$. For f_2, define $f_2(3) = 1$; $f_2(2) = 3$; and $f_2(1) = 4$. Then g_2 must be $g_2(4) = 1$; $g_2(3) = g_2(2) = 2$; and $g_2(1) = 3$. We have $(f_1, g_1) \preceq (f_2, g_2)$, i.e., for each $p \in P$, $g_2 f_2(p) \leq g_1 f_1(p)$ and $\mathscr{L}(f_2, P)$ refines $\mathscr{L}(f_1, P)$, but $f_2 g_2(2) = 3 \not\leq 2 = f_1 g_1(2)$ and $\mathscr{L}(g_2, Q)$ does not refine $\mathscr{L}(g_1, Q)$.

Given that $\preceq$ "works nicely" on partitions on P but seemingly unpredictably on partitions on Q, we could rename $\preceq$ to $\preceq_P$ and define another pre-order $\preceq_Q$ which we would define for $(f_1, g_1), (f_2, g_2) \in \mathscr{G}$ by $(f_1, g_1) \preceq_Q (f_2, g_2)$ if for each $q \in Q$, $f_2 g_2(q) \sqsubseteq f_1 g_1(q)$.

Further, we could define $\preceq_{PQ}$ by $(f_1, g_1) \preceq_{PQ} (f_2, g_2)$ if and only if $(f_1, g_1) \preceq_P (f_2, g_2)$ and $(f_1, g_1) \preceq_Q (f_2, g_2)$. It would seem that $\preceq_{PQ}$ might have promise for FCA use because it refines partitions in both $\wp(G)$ and $\wp(M)$.

The next definition comes from Ore [13].

Definition 2.13 Let (f, P, Q, g) be a Galois connection. (f, g) is said to be *perfect* if each element in P is a node and each element in Q is a node.

Proposition 2.14 If (f, g) is a perfect Galois connection, then (f, g) is a maximal element in $(\mathscr{G}, \preceq_{PQ})$.

In the introduction, we mention that studying FCA can help generate new Galois connection results. FCA motivated the defining of $\preceq_P$ and then of $\preceq_Q$ and $\preceq_{PQ}$.

Given the symmetric of Galois connections in general and the symmetric of $\preceq_{PQ}$, one might think that $\preceq_{PQ}$ would be a partial order. However, as Example 2.15 shows, this is not the case.

Example 2.15 Let P be the four element set $\{\bot, a, b, \top\}$, and define the partial order $\leq$ on P such that $\bot$ is less than or equal to everything, everything is less than or equal to $\top$, and a and b are not related. Let $(Q, \leq) = (P, \leq)$, and define (f, g) and (f', g') such that $f = g$ where $f(\bot) = \top$, $f(\top) = \bot$, $f(a) = a$, and $f(b) = b$.

Let $f' = f$ except $f'(a) = b$ and $f'(b) = a$, and let $g' = f'$. Then $(f, g) \neq (f', g')$, but $(f, g) \preceq_{PQ} (f', g')$ and $(f', g') \preceq_{PQ} (f, g)$. Also, both (f, g) and (f', g') are maximal elements in $(\mathscr{G}, \preceq_{PQ})$. It is also true that $(P, \leq)$ is isomorphic to the powerset of a two element set.

2.3 Yet Another Ordering on $\mathscr{G}$

For this ordering, we need and use results from the next section. In particular, we use the category **Gal** and the fact that **Gal** is concrete over **Set** $\times$ **Set** with the forgetful functor

$$U_G : \mathbf{Gal} \to \mathbf{Set} \times \mathbf{Set}$$

defined by

$$U_G(f, (P, \leq), (Q, \sqsubseteq), g) = (P, Q),$$

where P and Q in the image ordered pair are sets without partial orderings.

From [1], we can define a pre-order on the fibers of U_G such that if $(f_1, (P_1, \leq_1), (Q_1, \sqsubseteq_1), g_1)$ and $(f_2, (P_2, \leq_2), (Q_2, \sqsubseteq_2), g_2)$ are **Gal**-objects with $U_G(f_1, g_1) = U_G(f_2, g_2)$, i.e., such that $P_1 = P_2$ and $Q_1 = Q_2$, then $(f_1, g_1) \sqsubseteq (f_2, g_2)$ if and only if the **Set** $\times$ **Set** identity $(1_P, 1_Q) : P \times P \to Q \times Q$ where $P = P_1 = P_2$ and $Q = Q_1 = Q_2$ is a **Gal**-morphism

$$(1_P, 1_Q) : (f_1, (P, \leq_1), (Q, \sqsubseteq_1), g_1) \to (f_2, (P, \leq_2), (Q, \sqsubseteq_2), g_2).$$

Using item 5 of Proposition 1.5, we know that

$$(f_1, (P, \leq_1), (Q, \sqsubseteq_1), g_1) \sqsubseteq (f_2, (P, \leq_2), (Q, \sqsubseteq_2), g_2)$$

if and only if the set of nodes in $(P, \leq_1)$ is a subset of the set of nodes of $(P, \leq_2)$ and the set of nodes of $(Q, \sqsubseteq_1)$ is a subset of the set of nodes of $(Q, \sqsubseteq_2)$ and $(1_P, 1_Q)$ is a **Gal**-morphism.

We have defined $\sqsubseteq$ in a categorical setting, and in fact, a fiber in U_G is more complex than $\mathscr{G}$. In $\mathscr{G}$, we are assuming that $(P, \leq)$ and $(Q, \sqsubseteq)$ are fixed. However, for a fiber of U_G, we only have P and Q fixed; the partial orders are not fixed. We let $\mathscr{H}$ be a fiber of U_G.

Remark 2.16 $(\mathscr{H}, \sqsubseteq)$ is an interesting pre-ordered set. We can define partial orders $\leq$ and $\sqsubseteq$ on P and Q, respectively, so that both have largest elements. Then we can define a Galois connection, $(f, (P, \leq), (Q, \sqsubseteq), g)$ such that f and g are constant functions where f maps everything in P to the largest element in $(Q, \sqsubseteq)$ and g maps everything in Q to the largest element in $(P, \leq)$. $(f, (P, \leq), (Q, \sqsubseteq), g)$ is a minimal element in $(\mathscr{H}, \sqsubseteq)$ though it may not be a least element.

If $|P| = |Q|$, then we can define anti-isomorphic partial orders $\leq$ and $\sqsubseteq$ on P and Q, respectively. If f is an anti-isomorphism and if $g = f^{-1}$, then $(f, (P, \leq), (Q, \sqsubseteq), g)$ is a maximal element in $(\mathscr{H}, \sqsubseteq)$ though it may not be a greatest element.

We use what we have learned from this ordering on $\mathscr{H}$ to define an ordering on $\mathscr{G}$.

Definition 2.17 Let $(P,\leq)$ and $(Q,\sqsubseteq)$ be partially ordered sets. Define $\sqsubseteq$ on $\mathscr{G}$ by

$$(f_1,(P,\leq),(Q,\sqsubseteq),g_1) \sqsubseteq (f_2,(P,\leq),(Q,\sqsubseteq),g_2)$$

if and only if

$$\mathscr{N}(f_1,P) \subset \mathscr{N}(f_2,P) \text{ and } \mathscr{N}(Q,g_1) \subset \mathscr{N}(Q,g_2).$$

Proposition 2.18 The ordered sets $(\mathscr{G},\preceq_{\mathscr{P}\mathscr{Q}})$ and $(\mathscr{G},\sqsubseteq)$ are the same.

Proof: Proposition 2.11. •

3 Categories of Formal Contexts and Categories of Galois Connections

We define and study two categories of formal contexts. These categories are interesting, in part, because, though the objects of these categories are formal contexts, the morphisms are pairs of maps between polarities, i.e., between Galois connections whose partially ordered sets are powersets of the formal context sets. This is actually natural because, as stated in Section 1, the usefulness of FCA comes from the Galois connections determined by the relations in the formal contexts. Thus, it is appropriate that the domains and codomains of the morphisms in these categories of formal contexts be Galois connections between the powersets of the formal contexts, i.e., the domains and codomains should be polarities.

The category **Gal** is given in [11]. In [6], a similar but different category with formal contexts as objects is defined. Though the work in [6] also builds on the results in [11], the category in [6] differs from the category defined below in that both h and k in the definition below go from components of the domain to components of the codomain. In [6], the function k is defined such that $k : Q_2 \to Q_1$. This surprising direction of k actually follows naturally when generalizing the topological systems work of S. J. Vickers [16] which was done in [5] and then used in [6] to build relations between formal contexts and systems.

Definition 3.1 **Gal** is the category whose objects are Galois connections, $(f,(P,\leq),(Q,\sqsubseteq),g)$ and whose morphisms

$$(h,k) : (f_1,(P_1,\leq_1),(Q_1,\sqsubseteq_1),g_1) \to (f_2,(P_2,\leq_2),(Q_2,\sqsubseteq_2),g_2)$$

are such that $h : P_1 \to P_2$ and $k : Q_1 \to Q_2$ are functions with $k \circ f_1 = f_2 \circ h$ and $h \circ g_1 = g_2 \circ k$.

The following proposition comes from [11].

Proposition 3.2 If $(f_1,(P_1,\leq_1),(Q_1,\sqsubseteq_1),g_1)$ and $(f_2,(P_2,\leq_2),(Q_2,\sqsubseteq_2),g_2)$ are **Gal**-objects and if $h : P_1 \to P_2$ and $k : Q_1 \to Q_2$ are functions, then the following are equivalent where the conjunction of the three conditions in item 2 is equivalent to items 1 and 3.

1. (h,k) is a **Gal**-morphism.
2. (a) h maps fixed points in P_1 to fixed points in P_2, and k maps fixed points in Q_1 to fixed points in Q_2.
   (b) h and k are level-preserving. Thus, if p and p' are in the same leaf or are equivalent in P_1, then $h(p)$ and $h(p')$ are in the same leaf in P_2, and if q and q' are in the same leaf of Q_1, then $k(q)$ and $k(q')$ are in the same leaf of Q_2.
   (c) If $p \in P_1$ and $q \in Q_1$ in anti-isomorphic leaves of (f_1,g_1), then $h(p)$ and $k(q)$ are also in anti-isomorphic leaves of (f_2,g_2).

3. If x and y are in the same leaf of P_1 (respectively, of Q_1), then they are mapped by either path to the same fixed point of Q_2 (respectively, of P_2).

Definition 3.3 **Pol** is the full subcategory of **Gal** such that a **Gal**-object is a **Pol**-object if and only if it is a polarity.

Definition 3.4 **FC** is the category whose objects are formal contexts and whose morphisms are those of **Pol**. That is, if $\mathscr{K}_1 := (G_1, M_1, R_1)$ and $\mathscr{K}_2 := (G_2, M_2, R_2)$ are formal contexts, then $(h,k) : \mathscr{K}_1 \to \mathscr{K}_2$ is a morphism in **FC** if and only if

$$(h,k) : (H_1, (\wp(G_1), \subset), (\wp(M_1), \subset), K_1) \to (H_2, (\wp(G_2), \subset), (\wp(M_2), \subset), K_2)$$

is a **Pol**-morphism.

Proposition 3.5 **FC** and **Pol** are isomorphic categories.

In [11], it is shown that **Gal** is complete, cocomplete, well-powered, and co-well-powered [1]. Thus, **Gal** is a structurally rich and well behaved. Since **Pol** is a full subcategory of **Gal** and **FC** is isomorphic to **Pol**, then **FC** is seemingly "close" to also being structurally rich. However, the constructions which make **Gal** well behaved may not be closed with respect to **Pol**, and thus, these constructions may not be applicable to **FC**. Though this is true, we can via an embedding essentially make the constructions closed with respect to **Pol** because, as shown below, each object in **Gal** can be embedded into some **Pol**-object. Thus, we can perform our constructions on objects in **Pol** by doing the construction in **Gal** and then embedding the results back into **Pol**. Thus, in a natural sense, we have these constructions in **FC**, and thus, **FC** is structurally rich and well behaved.

Given an arbitrary Galois connection, in Proposition 3.8 we define the **Gal**-morphism which embeds this arbitrary Galois connection into a polarity. In Definition 3.12, we give a precise definition of an embedding, and in Theorem 3.14, we prove that the morphism defined in Proposition 3.8 is indeed an embedding.

Construction 3.6 Let $(f, (P, \leq), (Q, \sqsubseteq), g)$ be a Galois connection. Define $F : \wp(P) \to \wp(Q)$ such that for $A \subset P$,

$$F(A) = \bigcap\nolimits_{p \in A} \downarrow f(p),$$

and likewise, define $G : \wp(Q) \to \wp(P)$ such that for $B \subset Q$,

$$G(B) = \bigcap\nolimits_{q \in B} \downarrow g(q).$$

Proposition 3.7 Let $(f, (P, \leq), (Q, \sqsubseteq), g)$ be a Galois connection. $(F, (\wp(P), \subset), (\wp(Q), \subset), G)$ is a polarity.

Proof: Clearly, F and G are order-reversing because as the argument increases, the corresponding intersection becomes smaller. Let $A \subset P$, and let $p \in A$. $F(A) \subset \downarrow f(p)$. Therefore, for each $q \in F(A)$, we have $q \sqsubseteq f(p)$. It follows for each $q \in F(A)$, that $p \leq g \circ f(p) \leq g(q)$, and thus, $p \in \downarrow g(q)$ for each $q \in F(A)$. Hence, $p \in G \circ F(A)$ for each $p \in A$, and therefore, $A \subset G \circ F(A)$. Similarly, $1_{\wp(Q)} \subset F \circ G$. •

Proposition 3.8 Let $(f, (P, \leq), (Q, \sqsubseteq), g)$ be a Galois connection. Define $i_P : P \to \wp(P)$ such that $i_P(p) = \downarrow p$, for each $p \in P$. Likewise, define $i_Q : Q \to \wp(Q)$ by $i_Q(q) = \downarrow q$, for each $q \in Q$.

$$(i_P, i_Q) : (f, (P, \leq), (Q, \sqsubseteq), g) \to (F, (\wp(P), \subset), (\wp(Q), \subset), G)$$

is a **Gal**-morphism. Additionally, (i_P, i_Q) is a monomorphism in **Gal**.

Proof: Let $s \in P$. It follows that $i_P(s) = \downarrow s$. If $p \leq s$, i.e., if $p \in \downarrow s$, then since f is order-reversing, $\downarrow f(s) \subset \downarrow f(p)$. Hence,

$$F(\downarrow p) = \downarrow f(p) = i_Q(f(p)).$$

Hence, $Fi_P = i_Q f$. Similarly, $i_P g = i_Q f$. Thus, (i_P, i_Q) is a **Gal**-morphism.

From [11], we know that (i_P, i_Q) is a monomorphism in **Gal** if and only if both i_P and i_Q are injections, and since $\leq$ and $\sqsubseteq$ are both partial orders, then i_P and i_Q are both injections. •

Given a Galois connection $(f, (P, \leq), (Q, \sqsubseteq), g)$, we specify the polarity $(F, (\wp(P), \subset), (\wp(Q), \subset), G)$ by defining the order-reversing maps F and G. We could have specified the polarity by defining the appropriate relation from P to Q. From Construction 2.8, we know that this relation $R \subset P \times Q$ is $(p,q) \in R$ if and only if $q \in F(\{p\})$. Thus, $(p,q) \in R$ if and only if $q \in \downarrow f(p)$. By the alternate definition of a Galois connection, Definition 1.3,

$$(p,q) \in R \text{ if and only if } p \leq g(q) \text{ and } q \sqsubseteq f(p).$$

The following four definitions come from [1].

Definition 3.9 Let $L : \mathbf{X} \to \mathbf{Y}$ be a functor. L is *faithful* if whenever $m, n : X_1 \to X_2$ are distinct morphisms in $\mathbf{X}$, then $L(m)$ and $L(n)$ are distinct morphisms in $\mathbf{Y}$. Said differently, L is faithful if it is injective on the set of morphisms between each two objects in $\mathbf{X}$.

Definition 3.10 Let $\mathbf{X}$ be a category. A pair $(\mathbf{A}, U)$ is a *concrete category over* $\mathbf{X}$ if $\mathbf{A}$ is a category and if $U : \mathbf{A} \to \mathbf{X}$ is a faithful functor.

Definition 3.11 Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$. An $\mathbf{A}$-morphism $f : A \to B$ is an *initial morphism* if for any $\mathbf{A}$-object C, an $\mathbf{X}$-morphism $g : U(C) \to U(A)$ is an $\mathbf{A}$-morphism $g : C \to A$ whenever $f \circ g : C \to B$ is an $\mathbf{A}$-morphism.

Definition 3.12 Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$. If $f : A \to B$ is an initial $\mathbf{A}$-morphism and if $f : U(A) \to U(B)$ is a monomorphism in $\mathbf{X}$, then $f : A \to B$ is an *embedding*.

In the next proposition, **Set** is the category of sets and functions. The only part of this proposition which we need for our embedding result is the part involving $(\mathbf{Gal}, U_G)$.

Proposition 3.13 $(\mathbf{Gal}, U_G)$, $(\mathbf{Pol}, U_P)$, and $(\mathbf{FC}, U_F)$ are concrete over $\mathbf{Set} \times \mathbf{Set}$ where

$$U_G(f, (P, \leq), (Q, \sqsubseteq), g) = (P, Q),$$

$$U_P(H, \wp(G), \wp(M), K) = (\wp(G), \wp(M)), \text{ and}$$

$$U_F(G, M, R) = (\wp(G), \wp(M)).$$

Theorem 3.14 *Let $(f, (P, \leq), (Q, \sqsubseteq), g)$ be a Galois connection. The* **Gal**-*morphism* $(i_P, i_Q) : (f, g) \to (F, G)$ *is an embedding.*

Proof: From Proposition 3.8, we know that (i_P, i_Q) is a monomorphism. Thus, we only need to show that it is initial. Let $(f_C, (P_C, \leq), (Q_C, \sqsubseteq), g_C)$ be a Galois connection, and let $(r, s) : (P_C, Q_C) \to (P, Q)$ be a morphism in $\mathbf{Set} \times \mathbf{Set}$ (i.e., r and s are set functions) such that

$$(i_P s, i_Q t) : (f_C, g_C) \to (F, G)$$

is a **Gal**-morphism. It follows that

$$i_Q t f_C = F i_P s = i_Q f s.$$

Since i_Q is injective, then $t f_C = f s$. Similarly, $s g_C = g t$. Therefore, (r, s) is a **Gal**-morphism, and (i_P, i_Q) is initial. •

4 Example

The major vision or goal driving the development of the semantic web is to create a web which can be understood by computers with minimal intervention by humans. A major component of the semantic web is RDF, resource description framework. RDF stores information as triples where each triple is composed of a subject, a predicate, and an object. These triples are simple; each triple holds only an elementary amount of information. However, these triples can be combined to form large graphs representing significant and complex information.[2]

To help understand the information in a large RDF graph, it is helpful to have a schema of the information. The schema is a framework around which the information may be organized. Typically, elements of a schema are classes of subjects, and the schema may be organized with classes being subsets of other classes. Not surprisingly, a schema may be developed from information in the RDF graph itself. One way of creating a schema is to begin with the first two components of the RDF triples. These two components form a binary tuple consisting of a subject, which in FCA terms is an object, and a predicate or property. Each subject or object may have many associated properties, and multiple objects may have the same property. Thus, these binary subject-property tuples form a relation between the set of subjects or objects and the set of predicates.

The classes of a schema may be, at least, partially determined by the properties or predicates which the objects in the classes satisfy. Said differently, the classes of interest in forming the schema may be related to the object set nodes determined by the Galois connection or polarity of a formal context.

The semantic web is, however, not a static entity. It is dynamic; the RDF triples and the RDF graphs may change often. Thus, the corresponding schemas will, at times, need to change. How will changes in the RDF tuples affect the schema? Said differently, how will changes in the formal contexts change the schemas? This question is a motivation for this paper.

5 Conclusion

Our work raises questions about Galois connections and about formal concept analysis. Though in some situations, there may be good reasons for wanting to replace a formal context with a better formal context and though we have shown orderings which mathematically allow us to determine “better” formal contexts, this question must addressed from the perspective of FCA, i.e., from the perspective of FCA what criteria should be used to decide when one formal context is better than another? Though we can mathematically define better formal contexts, what in practice determines when one formal context is better than another?

When defining orderings on formal contexts, we have restricted ourselves to sets of formal contexts which have the same set of objects and the same set of properties; it may be useful to think about “better” formal contexts when the underlying sets of objects and/or properties may change. For example, when working with the Semantic Web, we will likely need to enlarge our set of objects. Can we make these changes and preserve, at least, in part the schema structure which is already in place?

In addition to understanding what makes one formal context better than another, future work may also include understanding what the categorical constructions in **FC** mean in practice. In [11], in addition to defining the category **Gal**, the category $\mathbf{Gal_p}$ is defined. $\mathbf{Gal_p}$ is a subcategory of **Gal** such that (h,k) is

[2]Thanks to S. Ayvaz and M. Aydar of Kent State University for explaining this example.

a $\mathbf{Gal_p}$-morphism if and only if it is a **Gal**-morphism and both h and k are order-preserving. It may be worthwhile to study the corresponding formal context category $\mathbf{FC_p}$.

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