Standardization in resource lambda-calculus

Maurizio Dominici Simona Ronchi Della Rocca
Dipartimento di Informatica – Università di Torino
dominicimaurizio@gmail.com ronchi@di.unito.it
Paolo Tranquilli
Dipartimento di Scienze dell'Informazione – Università di Bologna
tranquil@cs.unibo.it

The resource calculus is an extension of the λ-calculus allowing to model resource consumption. It is intrinsically non-deterministic and has two general notions of reduction – one parallel, preserving all the possible results as a formal sum, and one non-deterministic, performing an exclusive choice at every step. We prove that the non-deterministic reduction enjoys a notion of standardization, which is the natural extension with respect to the similar one in classical λ-calculus. The full parallel reduction only enjoys a weaker notion of standardization instead. The result allows an operational characterization of may-solvability, which has been introduced and already characterized (from the syntactical and logical points of view) by Pagani and Ronchi Della Rocca.

1 Introduction

The resource calculus (Λ^r) is an extension of the λ-calculus allowing to model resource consumption. Namely, the argument of a function comes as a finite multiset of resources, which in turn can be either linear or reusable. A linear resource must be used exactly once, while a reusable one can be called *ad libitum*. In this setting the evaluation of a function applied to a multiset of resources gives rise to different possible choices, because of the different possibilities of distributing the resources among the occurrences of the formal parameter. We can define two kinds of reduction, according to the interpretation we want to give to this fact. The parallel reduction (which can be further divided in giant and baby) performs all the possible choices, and gives as result a formal sum preserving all the possible results, while the non-deterministic reduction at every step chooses non-deterministically one of the possible results. In case of a multiset of linear resources, also a notion of *crash* arises, whenever the cardinality of the multiset does not fit exactly the number of occurrences. Then the resource calculus is a useful framework for studying the notions of linearity and non-determinism, and the relation between them. Λ^r is a descendant of the calculus of multiplicities, introduced by Boudol in [2], and it has been designed by Tranquilli [11] in order to give a precise syntax for the differential λ-calculus of Ehrhard and Regnier [4]. Λ^r can be used as a paradigmatic language for different kinds of computation. Usual λ-calculus can be embedded in it. Forbidding linear terms but allowing non-empty finite multisets of reusable terms yields a purely non-deterministic extension of λ-calculus, which recalls the one of De Liguoro and Piperno [3]. Allowing only multisets of linear terms gives the linear fragment of Λ^r, used by Ehrhard and Regnier to give a quantitative account to λ-calculus β-reduction through Taylor expansion [5, 6].

But to be effectively used, Λ^r needs a clear operational semantics. In this paper we investigate the notion of standardization in it. Let us recall that a calculus has the standardization property when every reduction sequence can be rearranged according to a predefined order between redexes. Namely a reduction is standard with respect to a given order if at every reduction step the reduced redex is not

S.Alves and I. Mackie (Eds.): Second International Workshop on Linearity 2012 (LINEARITY'12).
EPTCS 101, 2012, pp. 1–11, doi:10.4204/EPTCS.101.1

© M. Dominici & S. Ronchi Della Rocca & P. Tranquilli
This work is licensed under the
Creative Commons Attribution License.

Λ^r :	$M,N,L,O ::= x \mid \lambda x.M \mid MP$		terms
$\Lambda^{(!)}$:	$M^{(!)},N^{(!)}$	$::= M \mid M^!$	resources
Λ^b :	P,Q,R	$::= 1 \mid [M^{(!)}]\cdot P$	bags
$\Lambda^{(b)}$:	A,B	$::= M \mid P$	expressions
$\mathtt{Nat}\langle\Lambda^r\rangle$:	$\mathbb{M},\mathbb{N},\mathbb{L}$	$::= 0 \mid M \mid \mathbb{M}+\mathbb{N}$	sums of terms
$\mathtt{Nat}\langle\Lambda^b\rangle$:	$\mathbb{P},\mathbb{Q},\mathbb{R}$	$::= 0 \mid P \mid \mathbb{P}+\mathbb{Q}$	sums of bags
$\mathbb{A},\mathbb{B} \in \mathtt{Nat}\langle\Lambda^{(b)}\rangle := \mathtt{Nat}\langle\Lambda^r\rangle \cup \mathtt{Nat}\langle\Lambda^b\rangle$			sums of expressions

(a) Grammar of terms, bags, sums, expressions.

$$\lambda x.(\sum_i M_i) := \sum_i \lambda x.M_i$$
$$(\sum_i M_i)(\sum_j P_j) := \sum_{ij} M_i P_j$$
$$[(\sum_i M_i)]\cdot(\sum_j P_j) := \sum_{ij} [M_i]\cdot P_j$$
$$[(\sum_i^k M_i)^!]\cdot(\sum_j P_j) := \sum_j [M_1^!,\dots,M_k^!]\cdot P_j$$

(b) Notation on $\mathtt{Nat}\langle\Lambda^{(b)}\rangle$.

Figure 1: Syntax of the resource calculus.

a residual of a redex which, in the given order, precedes a previously reduced one. In the case of λ-calculus, the standardization is based on the left-to-right order of redexes.

In Λ^r, as the elements of a multiset are not ordered, a notion of standardization would be based on a partial order between redexes. A first result, corresponding to a weak notion of standardization, has been proved by Pagani and Tranquilli [9], stating that the reductions of redexes inside reusable resources can always be postponed. We define a stronger partial order between redexes, and we prove that the non-deterministic reduction enjoys the standardization property with respect to it. Even though this order is not total, it is in fact undefined if and only if the two redexes live in different elements of a same multiset, so that any finer order would not be well-defined. This result allows us to complete the characterization of may-solvability, defined in [8]. Let us stress that solvability is a key notion for evaluation, since it identifies the meaningful programs, and a clear notion of output result of a computation. Since this calculus is non-deterministic, two different notions of solvability arise, one optimistic (angelical, may) and one pessimistic (demoniac, must). In particular, in [8, 7] a characterization of the may-solvability has been given, from a syntactical and logical point of view. Here we provide an operational characterization, through an abstract reduction machine, performing the non-deterministic reduction. The soundness and completeness of the machine with respect to the notion of may-solvability comes from the standardization property.

Moreover we prove that the parallel reduction does not enjoy the same standardization property. Namely we show that in this case any order between linear redexes cannot be sound. This negative result is interesting, since it gives evidence to the deep difference between linear and non-deterministic reduction.

2 Syntax

The syntax of Λ^r. Basically, we have three syntactical sorts: terms, that are in functional position, bags, that are in argument position and represent multisets of resources, and finite formal sums, that represent the possible results of a computation. Precisely, Figure 1(a) gives the grammar for generating the set Λ^r of **terms** and the set Λ^b of **bags** (which are in fact finite multisets of **resources** $\Lambda^{(!)}$) together with their typical metavariables. A resource can be linear (it must be used exactly once) or not (it can be used ad libitum, also zero times), in the last case it is written with a ! superscript. Bags are multisets presented in multiplicative notation, so that $P\cdot Q$ is the multiset union, and $1 = []$ is the empty bag: that means, $P\cdot 1 = P$ and $P\cdot Q = Q\cdot P$. It must be noted though that we will never omit the dot $\cdot$, to avoid confusion

$$y\langle N/x\rangle := \begin{cases} N & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases} \qquad \begin{aligned} (\lambda y.M)\langle N/x\rangle &:= \lambda y.M\langle N/x\rangle, \\ (MP)\langle N/x\rangle &:= M\langle N/x\rangle P + MP\langle N/x\rangle, \end{aligned} \qquad \begin{aligned} 1\langle N/x\rangle &:= 0, \\ ([M]\cdot P)\langle N/x\rangle &:= [M\langle N/x\rangle]\cdot P + [M]\cdot P\langle N/x\rangle, \\ ([M^!]\cdot P)\langle N/x\rangle &:= [M\langle N/x\rangle, M^!]\cdot P + [M^!]\cdot P\langle N/x\rangle. \end{aligned}$$

Figure 2: Linear substitution. In the abstraction case we suppose $y \notin \mathrm{FV}(N)\cup\{x\}$.

with application. **Sums** are multisets in additive notation, with 0 referring to the empty multiset, so that: $\mathbb{M}+0 = \mathbb{M}$ and $\mathbb{M}+\mathbb{N} = \mathbb{N}+\mathbb{M}$. We use two different notations for multisets in order to underline the different role of bags and sums.

An **expression** (whose set is denoted by $\Lambda^{(b)}$) is either a term or a bag. Though in practice only sums of terms are needed, for the sake of the proofs we also introduce sums of bags and of expressions. The symbol $\mathtt{Nat}$ denotes the set of natural numbers, and $\mathtt{Nat}\langle\Lambda^r\rangle$ (resp. $\mathtt{Nat}\langle\Lambda^b\rangle$) denotes the set of finite formal sums of terms (resp. bags).

The grammar for terms and bags does not include sums in any point, so that in a sense they may arise only as a top level constructor. However, as an inductive notation (and *not* in the actual syntax) we extend all the constructors to sums as shown in Figure 1(b). In fact, all constructors but the $(\cdot)^!$ are, as expected, linear in the algebraic sense, i.e. they commute with sums. In particular, we have that 0 is always absorbing but for the $(\cdot)^!$ constructor, in which case we have $[0^!] = 1$. We refer to [11, 10] for the mathematical intuitions underlying the resource calculus.

We adopt α-equivalence and all the usual λ-calculus conventions as per [1].

The pair reusable/linear has a counterpart in the following two different notions of substitutions: their definition, hence that of reduction, heavily uses the notation of Figure 1(b).

Definition 1 (Substitutions)**.** We define the following substitution operations.

(i) $A\{N/x\}$ is the usual λ-calculus (*i.e.* capture free) substitution of N for x. It is extended to sums as in $\mathbb{A}\{\mathbb{N}/x\}$ by linearity in $\mathbb{A}$. The form $A\{x+N/x\}$ is called **partial substitution**.

(ii) $A\langle N/x\rangle$ is the **linear substitution** defined inductively in Figure 2. It is extended to $\mathbb{A}\langle\mathbb{N}/x\rangle$ by bilinearity in both $\mathbb{A}$ and $\mathbb{N}$.

(iii) $A\langle\langle N^{(!)}/x\rangle\rangle$, defined by $A\langle\langle N/x\rangle\rangle := A\langle N/x\rangle$ and $A\langle\langle N^!/x\rangle\rangle := A\{N+x/x\}$, is the **resource substitution**, and moreover $A\langle\langle B/x\rangle\rangle$, defined by $A\langle\langle [N_1^{(!)},\dots,N_n^{(!)}]/x\rangle\rangle = A\langle\langle N_1^{(!)}/x\rangle\rangle\cdots\langle\langle N_n^{(!)}/x\rangle\rangle$ (assuming $x \notin FV(B)$) is the **bag substitution**.

Roughly speaking, the linear substitution corresponds to the replacement of the resource to exactly one *linear* occurrence of the variable. In the presence of multiple occurrences, all the possible choices are made, and the result is the sum of them. For example $(y[x][x])\langle N/x\rangle = y[N][x]+y[x][N]$. In the case there are no free linear occurrences, then linear substitution returns 0, morally an error message. For example $(\lambda y.y)\langle N/x\rangle = \lambda y.(y\langle N/x\rangle) = \lambda y.0 = 0$. Finally, in case of reusable occurrences of the variable, linear substitution acts on a linear copy of the variable, *e.g.* $[x^!]\langle N/x\rangle = [N, x^!]$.

The reductions of Λ^r. A term context $C[\![\cdot]\!]$ (or a bag context $P[\![\cdot]\!]$) is defined by extending the syntax of terms and bags by a distinguished free variable called **hole** and denoted by $[\![\cdot]\!]$.

Notice that in contexts the order of holes cannot be truly established as bags are independent of order. So filling[1] the k holes of a contexts by terms needs a bijective mapping a from $\{1,\dots,k\}$ to hole

[1] We recall that hole substitution allows for variable capture.

occurrences in $C[\![\cdot]\!]$, and $C_a[\![\vec{M_i}]\!]$ denotes the replacement of the holes by $M_1,\dots,M_k$ guided by this map. We can write also $C[\![\vec{M_i}]\!]$, by considering an implicit map.

A (term, bag) context is **simple** if it contains exactly one occurrence of the hole. In this case we will write simply $C[\![M]\!]$ for the result of filling of the hole with M. A simple context is **linear** if its hole is not under the scope of a $()^!$ operator, and it is **applicative** if it has the hole not in a bag. As usual the (simple/applicative/linear) context closure of a relation R is the one relating $C[\![t]\!]$ and $C[\![t']\!]$ when $t\,R\,t'$ and C is of the appropriate kind.

We define two kinds of reduction rule, called parallel and non-deterministic. Moreover the parallel reduction can be further divided into baby-step and giant-step, the former being a decomposition of the latter. Baby-step is more atomic, performing one substitution at a time, while the giant-step is closer to λ-calculus β-reduction, wholly consuming its redex in one shot.

Definition 2 ([11, 10])**.** (i) The **parallel** reductions are defined as follows:

- The **baby-step** reduction $\xrightarrow{\mathtt{b}}$ is defined by the simple context closure of the following relation (assuming x not free in N):

$$(\lambda x.M)1 \xrightarrow{\mathtt{b}} M\{0/x\} \qquad (\lambda x.M)[N]\cdot P \xrightarrow{\mathtt{b}} (\lambda x.M\langle N/x\rangle)P$$
$$(\lambda x.M)[N^!]\cdot P \xrightarrow{\mathtt{b}} (\lambda x.M\{N+x/x\})P$$

- The **giant-step** reduction $\xrightarrow{\mathtt{g}}$ is defined by the simple context closure of the following relation:

$$(\lambda x.M)P \xrightarrow{\mathtt{g}} M\langle\!\langle P/x\rangle\!\rangle\{0/x\}$$

(ii) The **non-deterministic** reduction is the relation $M \xrightarrow{\mathtt{nd}} N$ if and only if $M \xrightarrow{\mathtt{g}} N + \mathbb{A}$ for some $\mathbb{A}$.

Notation 3. For any reduction $\xrightarrow{\varepsilon}$ (the ones listed above and the ones to come), we denote by $\xrightarrow{\varepsilon *}$ its reflexive-transitive closure. $\rho : M \xrightarrow{\varepsilon *} N$ denotes a particular reduction sequence from M to N, and $|\rho|$ its length.

Λ^r and λ-calculus. In λ-calculus, arguments can be used as many times we want, so it is easy to inject it in Λ^r through the following translation $(.)^*$:

$$(x)^* = x,\;\; (\lambda x.M)^* = \lambda x.(M)^*,\;\; (MN)^* = (M)^*[(N)^{*!}]$$

On terms of Λ^r which are translations of λ-terms, the giant reduction becomes the usual β-reduction.

3 Standardization

In this section we will prove that the non-deterministic reduction enjoys a standardization property. As we recalled already in the introduction, the standardization property is based on an order relation between redexes. We can define it formally as follows:

Definition 4. Let $\prec$ be an order on positions in terms (which is extended to an order on subterms of a given term). Suppose ρ is a reduction chain, and let M_i and R_i be the i-th term and fired redex in ρ respectively. We say that ρ is $\prec$-*standard* if for every i we have that R_{i+1} is not the residual of a redex R' in M_i such that $R' \prec R_i$.

We will prove that non deterministic reduction in Λ^r enjoys the standardization property with respect to the order $\prec_r$, which is the partial order on positions in Λ^r terms that, intuitively, gives precedence to linear positions over non-linear ones, and then orders linear positions left-to-right, with the proviso that positions inside the same bag be not comparable. The formal definition follows.

Definition 5 (Linear left-to-right order)**.** For two subterms S_1 and S_2 inside the expression $\mathbb{A}$, we say that $S_1 \prec_r S_2$ in $\mathbb{A}$ if and only if any of the following happens:

- S_2 is a subterm of S_1;
- S_1 is linear in $\mathbb{A}$ while S_2 is not;
- S_1 and S_2 are both linear in $\mathbb{A}$, $\mathbb{A} = MP$, S_1 is in M and S_2 is in P.
- S_1 and S_2 are subterms of the same proper subexpression $\mathbb{B}$ of $\mathbb{A}$, and $S_1 \prec_r S_2$ in $\mathbb{B}$;

Example 6. $S_1 \prec_r S_2$ in both $\lambda x.x[S_2^!][S_1]$ and $\lambda x.x[S_1][S_2]$, while they are incomparable in $\lambda x.x[S_1, S_2]$.

Our starting point is the division of redexes in two classes, outer and inner.

Definition 7 ([9])**.** Let $\varepsilon \in \{\mathtt{b}, \mathtt{g}, \mathtt{nd}\}$. The **outer ε-reduction** $\xrightarrow{\mathtt{o}\varepsilon}$ is the *linear* context closure of the ε-steps given in Definitions 2. A **non-outer ε-reduction**, called **inner** is defnoted by $\xrightarrow{\mathtt{i}\varepsilon}$.

In other words, an outer reduction does not reduce inside reusable resources, so an outer redex (*i.e.* a redex for $\xrightarrow{\mathtt{o}\varepsilon}$) is a redex not under the scope of a $(\cdot)^!$ constructor. In particular a term corresponding to a λ-term has at most one outer-redex, which coincides with the head-redex. Pagani and Tranquilli stated in some sense a weak standardization property for the giant reduction, proving that inner redexes can always be postponed. Their result can easily be extended to other reductions, in particular to the non-deterministic one.

Theorem 8 ([9])**.** *Let* $\varepsilon \in \{\mathtt{b}, \mathtt{g}, \mathtt{nd}\}$. $M \xrightarrow{\varepsilon^*} \mathbb{A}$ *implies* $M \xrightarrow{\mathtt{o}\varepsilon^*} \mathbb{A}'$ *and* $\mathbb{A}' \xrightarrow{\mathtt{i}\varepsilon^*} \mathbb{A}$.

We introduce now a further classification between outer redexes.

Definition 9. The set of *leftmost* redexes $\mathscr{L}(M)$ of a term M or a bag P are defined inductively by:

$$\begin{array}{c} \mathscr{L}(x) := \emptyset, \\ \mathscr{L}(\lambda x.M) := \mathscr{L}(M) \end{array} \quad \mathscr{L}(MP) := \begin{cases} \{MP\} & \text{if } M = \lambda x.M', \\ \mathscr{L}(M) & \text{otherwise, if } \mathscr{L}(M) \neq \emptyset \\ \mathscr{L}(P) & \text{otherwise} \end{cases} \quad \begin{array}{c} \mathscr{L}(1) := \emptyset, \\ \mathscr{L}([M^!] \cdot P := \mathscr{L}(P), \\ \mathscr{L}([M] \cdot P) := \mathscr{L}(M) \cup \mathscr{L}(P) \end{array}$$

In regular λ-calculus, the set $\mathscr{L}(M)$ is at most a singleton, and $\prec_r$-standardness collapses to the regular notion of left-to-right order of redexes.

Fact 10. Redexes in $\mathscr{L}(M)$ are exactly the $\prec_r$-minimal elements among all redexes of M.

In the following, we will consider in particular the non-deterministic reduction. So, let us introduce some notation.

Notation 11. Let $M \xrightarrow{\mathtt{ndo}} N$. $M \xrightarrow{\mathtt{lm}} N$ denotes that the reduction fires a redex in $\mathscr{L}(M)$, while we write $M \xrightarrow{\neg\mathtt{lm}} N$ if the redex is not a leftmost one. Moreover $M \xrightarrow{\mathtt{o}} N$ and $M \xrightarrow{\mathtt{i}} N$ will be short for for $M \xrightarrow{\mathtt{ndo}} N$ and $M \xrightarrow{\mathtt{ndi}} N$ respectively.

Lemma 12. *We have the following facts on non-leftmost reduction.*

- $\rho : \lambda x.M \xrightarrow{\neg\mathtt{lm}*} N$ *if and only if* $N = \lambda x.M'$ *and* $\rho' : M \xrightarrow{\neg\mathtt{lm}*} M'$ *with* $|\rho| = |\rho'|$;
- $\rho : MP \xrightarrow{\neg\mathtt{lm}*} N$ *if and only if* $N = M'P'$, $\rho' : M \xrightarrow{\neg\mathtt{lm}*} M'$ *and* $\rho'' : P \xrightarrow{\mathtt{o}*} P'$ *with* $|\rho| = |\rho'| + |\rho''|$;

- $\rho : [M] \cdot P \xrightarrow{\neg \mathtt{lm}*} Q$ *if and only if* $Q = [M'] \cdot P'$, $\rho' : M \xrightarrow{\neg \mathtt{lm}*} M'$ *and* $\rho'' : P \xrightarrow{\neg \mathtt{lm}*} P'$ *with* $|\rho| = |\rho'| + |\rho''|$*;*
- $\rho : [M^!] \cdot P \xrightarrow{\neg \mathtt{lm}*} Q$ *if and only if* $Q = [M^!] \cdot P'$ *and* $\rho'' : P \xrightarrow{\neg \mathtt{lm}*} P'$ *with* $|\rho| = |\rho''|$.

The proof of standardization is based on an inversion property between outer redexes, saying that a not-leftmost reduction followed by a leftmost one can always be replaced by a leftmost followed by an outer. This is the upcoming Lemma 15. In order to get it we first prove the following intermediate properties.

Lemma 13. *If* $O \xrightarrow{\circ} O'$ *then* $\forall L' \in O'\langle\langle Q/x\rangle\rangle \{0/x\} \, \exists L \in O\langle\langle Q/x\rangle\rangle \{0/x\}$ *such that* $L \xrightarrow{\circ} L'$.

Proof. We will prove that $\forall L' \in O'\langle\langle Q/x\rangle\rangle \exists L \in O\langle\langle Q/x\rangle\rangle$ such that $L \xrightarrow{\circ} L'$. Then the statement of the lemma follows easily. By induction on O.

Case 1. $O = x$ and $O = y$ are not possible.

Case 2. $O = \lambda y.M$. By inductive hypothesis.

Case 3. $O = (\lambda y.M)P$. There are three cases: $\lambda y.M \xrightarrow{\circ} \lambda y.M'$, $P \xrightarrow{\circ} P'$, $(\lambda y.M)P \xrightarrow{\circ} O' \in M\langle\langle P/y\rangle\rangle \{0/y\}$. Let $M \xrightarrow{\circ} M'$. $(\lambda y.M)P\langle\langle Q/x\rangle\rangle = \sum_{Q_1,Q_2}((\lambda y.M)\langle\langle Q_1/x\rangle\rangle)(P\langle\langle Q_2/x\rangle\rangle)$,where Q_1, Q_2 range over all the possible decomposition of Q into two parts, counting the reusable resources with all the possible multiplicities. This means that in case Q_1, Q_2 are considered two different subterms also in case they are syntactically equal. By inductive hypothesis, for all $L' \in (\lambda y.M')\langle\langle Q_1/x\rangle\rangle$ there is $L \in (\lambda y.M)\langle\langle Q_1/x\rangle\rangle$ such that $L \xrightarrow{\circ} L'$, and the result follows by transitivity of $\in$. The case $P \xrightarrow{\circ} P'$ is similar.

Let $(\lambda y.M)P \xrightarrow{\circ} O' \in M\langle\langle P/y\rangle\rangle \{0/y\}$. Then we have that the substitution $(\lambda y.M)P\langle\langle Q/x\rangle\rangle$ is equal to the sum $\sum_{Q_1,Q_2}(\lambda y.M\langle\langle Q_1/x\rangle\rangle)(P\langle\langle Q_2/x\rangle\rangle)$, where Q_1, Q_2 range as before. Since each component of this sum is a redex (the substitutions do not modify the external shape of the terms), we can reduce each redex, so obtaining that for all $L \in (\lambda y.M\langle\langle Q_1/x\rangle\rangle)(P\langle\langle Q_2/x\rangle\rangle)$, $L \xrightarrow{\circ} L' \in M\langle\langle Q_1/x\rangle\rangle\langle\langle P\langle\langle Q_2/x\rangle\rangle/y\rangle\rangle \{0/y\}$. On the other side, $M\langle\langle P/y\rangle\rangle \{0/y\} \langle\langle Q/x\rangle\rangle$ is equal to the sum $\sum_{Q_1,Q_2} M\langle\langle Q_1/x\rangle\rangle\langle\langle P\langle\langle Q_2/x\rangle\rangle/y\rangle\rangle \{0/y\}$, and the proof is done.

Case 4. $O = MP$ and $O' = M'P$ or $O = MP$ and $O' = MP'$. All by inductive hypothesis. □

Lemma 14. *If* $Q \xrightarrow{\circ} Q'$ *then* $\forall L' \in O\langle\langle Q'/x\rangle\rangle \{0/x\} \, \exists L \in O\langle\langle Q/x\rangle\rangle \{0/x\}$ *such that* $L \xrightarrow{\circ} L'$.

Proof. By induction on Q. Q cannot be 1 as it would be normal.

If $Q = [H] \cdot P$ then $O\langle\langle Q/x\rangle\rangle \{0/x\} = O\langle H/x\rangle\langle\langle P/x\rangle\rangle \{0/x\}$. We proceed by cases:

Case 1. The reduction is on P, *i.e.* $[H] \cdot P \xrightarrow{\circ} [H] \cdot P'$. For all $N \in O\langle H/x\rangle$, $N\langle\langle P/x\rangle\rangle \xrightarrow{\circ} N\langle\langle P'/x\rangle\rangle$. Then we have by induction that for all $L \in N\langle\langle P/x\rangle\rangle \{0/x\}$ there is $L' \in N\langle\langle P'/x\rangle\rangle \{0/x\}$ such that $L \xrightarrow{\circ} L'$. So the result follows.

Case 2. The reduction is on H, *i.e.* $[H] \cdot P \xrightarrow{\circ} [H'] \cdot P$). Let us set $O\langle H/x\rangle\langle\langle P/x\rangle\rangle \{0/x\} = (O_1 + ... + O_k)\langle\langle P/x\rangle\rangle \{0/x\}$, where H occurs in all O_j $(1 \leq j \leq k)$, since the substitution is linear. Let $O_j \xrightarrow{\circ g} O^j_1 + ... + O^j_m$ by reducing the occurrence of H in it. So $O_j \xrightarrow{\circ} O^j_i$ $(1 \leq i \leq m_j)$, and, by Lemma 13, for all $L' \in O^j_i\langle\langle P/x\rangle\rangle \{0/x\}$, there is $L \in O_j\langle\langle P/x\rangle\rangle \{0/x\}$ such that $L \xrightarrow{\circ} L'$. Since $O_j \in O\langle H/x\rangle$ and $O^j_i \in O\langle H'/x\rangle$, the proof follows.

If $Q = [H^!] \cdot P$ the reduction on P, and the case is similar to the first case of the previous point. □

Lemma 15 (Inversion)**.** $M \xrightarrow{\neg \mathtt{lm}} M' \xrightarrow{\mathtt{lm}} N$ *implies* $M \xrightarrow{\mathtt{lm}} M'' \xrightarrow{\mathtt{ndo}} N$*, for some* M''.

Proof. We proceed by induction on M.
Let $M = \lambda\vec{y}.(\lambda x.O)QP_1...P_j...P_n$, so $\mathscr{L}(M) = \{(\lambda x.O)Q\}$. Non leftmost reductions on M can be done in O, in Q or in P_j $(1 \leq j \leq n)$. We procede by cases:

Case 1. The reduction is on $P_j(1 \leq j \leq n)$.]. Let $P_j \xrightarrow{\mathtt{g}} P'_j + \mathbb{S}$. We have that

$$M \xrightarrow{\neg\mathtt{lm}} \lambda\vec{y}.(\lambda x.O)QP_1...P'_j...P_n \xrightarrow{\mathtt{g}} \lambda\vec{y}.O\langle\langle Q/x\rangle\rangle\,\{0/x\}\,P_1...P'_j...P_n.$$

Moreover, by reducing the leftmost redex

$$M \xrightarrow{\mathtt{g}} \lambda\vec{y}.O\langle\langle Q/x\rangle\rangle\,\{0/x\}\,P_1...P_j...P_n = \lambda\vec{y}.(O_1+...+O_k)P_1...P_j...P_n,$$

so that

$$M \xrightarrow{\mathtt{lm}} \lambda\vec{y}.O_hP_1...P_j...P_n \xrightarrow{\mathtt{o}} \lambda\vec{y}.O_hP_1...P'_j...P_n$$

for all $1 \leq h \leq k$.

Case 2. The reduction is on Q. Let $Q \xrightarrow{\mathtt{o}} Q'$ and let $M \xrightarrow{\neg\mathtt{lm}} \lambda\vec{y}.(\lambda x.O)Q'P_1...P_n \xrightarrow{\mathtt{lm}} \lambda\vec{y}.\overline{M}P_1...P_n$, where $\overline{M}$ is such that $\overline{M} \in O\langle\langle Q'/x\rangle\rangle\,\{0/x\}$. Moreover, by reducing the leftmost redex, we also have the reduction $M \xrightarrow{\mathtt{g}} \lambda\vec{y}.O\langle\langle Q/x\rangle\rangle\,\{0/x\}\,P_1...P_n$. By Lemma 14, $Q \xrightarrow{\mathtt{o}} Q'$ implies that for all $L' \in O\langle\langle Q'/x\rangle\rangle\,\{0/x\}$, there exists $L \in O\langle\langle Q/x\rangle\rangle\,\{0/x\}$ such that $L \xrightarrow{\mathtt{o}} L'$. So there is $\overline{\overline{M}} \in O\langle\langle Q/x\rangle\rangle\,\{0/x\}$ such that $M \xrightarrow{\mathtt{lm}} \lambda\vec{y}.\overline{\overline{M}}P_1...P_n \xrightarrow{\mathtt{o}} \lambda\vec{y}.\overline{M}P_1...P_n$.

Case 3. The reduction is in O. Let $O \xrightarrow{\mathtt{o}} O'$, and let $M \xrightarrow{\neg\mathtt{lm}} \lambda\vec{y}.(\lambda x.O')QP_1...P_n \xrightarrow{\mathtt{lm}} \lambda\vec{y}.\overline{M}P_1...P_n$, where $\overline{M}$ is such that $\overline{M} \in O'\langle\langle Q/x\rangle\rangle\,\{0/x\}$. Again if we reduce the leftmost redex, we have the reduction $M \xrightarrow{\mathtt{g}} \lambda\vec{y}.O\langle\langle Q/x\rangle\rangle\,\{0/x\}\,P_1...P_n$. $O \xrightarrow{\mathtt{o}} O'$ implies, by Lemma 13, $\forall L' \in O'\langle\langle Q/x\rangle\rangle\,\{0/x\}\,, \exists L \in O\langle\langle Q/x\rangle\rangle\,\{0/x\}$ such that $L \xrightarrow{\mathtt{o}} L'$. So there is $\overline{\overline{M}} \in O\langle\langle Q/x\rangle\rangle\,\{0/x\}$ such that we can compose the reductions $M \xrightarrow{\mathtt{lm}} \lambda\vec{y}.\overline{\overline{M}}P_1...P_n \xrightarrow{\mathtt{o}} \lambda\vec{y}.\overline{M}P_1...P_n$.

Let $M = \lambda\vec{y}.xP_1...P_n$, and let $P_i \xrightarrow{\neg\mathtt{lm}} P'_i$ and $P_j \xrightarrow{\mathtt{lm}} P'_j$. In case $i \neq j$, the proof is trivial. In case $i = j$ the proof is by induction on P_i. □

Corollary 16. *If $\rho : M \xrightarrow{\mathtt{o}*} M'$ then there are $\sigma : M \xrightarrow{\mathtt{lm}*} M''$ and $\pi : M'' \xrightarrow{\neg\mathtt{lm}*} M'$ with $|\sigma| + |\pi| = |\rho|$.*

Lemma 17.

(i) *Given $\rho : M \xrightarrow{\mathtt{lm}*} N$ and $\sigma : N \xrightarrow{\mathtt{o}*} L$, then $\rho\sigma : M \xrightarrow{\mathtt{o}*} L$ is $\prec_r$-standard if and only if σ is. In particular every chain of leftmost reductions is $\prec_r$-standard.*

(ii) *Given $\rho : M \xrightarrow{\mathtt{o}*} N$ and $\sigma : N \xrightarrow{\mathtt{i}*} L$, then $\rho\sigma : M \xrightarrow{\mathtt{nd}*} L$ is $\prec_r$-standard if and only if both ρ and σ are.*

Proof. The result follows easily from the definition of $\prec_r$. □

Now we can prove that the non-deterministic outer reduction is $\prec_r$-standard.

Lemma 18 (Non-deterministic outer standard reduction)**.** *If $M \xrightarrow{\mathtt{o}*} N$, then there is a $\prec_r$-standard non-deterministic outer reduction from M to N.*

Proof. We reason by induction on the pair (p,s), where $p = |\rho|$ is the length of the reduction sequence $\rho : M \xrightarrow{\mathtt{o}*} N$, and s is the number of symbols in M. By Corollary 16, there is a reduction $\sigma_l : M \xrightarrow{\mathtt{lm}*} M'$ and $\sigma_r : M' \xrightarrow{\neg\mathtt{lm}*} N$ with $|\sigma_l| + |\sigma_r| = |\rho| = p$. If $|\sigma_l| > 0$ then inductive hypothesis applies to σ_r, giving $\prec_r$-standard $\sigma'_r : M' \xrightarrow{\mathtt{o}*} N$, which gives that $\sigma_l\sigma'_r : M \xrightarrow{\mathtt{o}*} N$ is $\prec_r$-standard by Lemma 17. In case

$\sigma_r : M \xrightarrow{\neg\mathtt{lm}*} N$ is the whole reduction, the proof is by cases on M. The only non-obvious case is when $M = LP$: by Lemma 12 we have $N = L'P'$ and $\rho' : L \xrightarrow{\neg\mathtt{lm}*} L'$ and $\rho'' : P \xrightarrow{\mathtt{o}*} P'$. We can apply inductive hypothesis to both as $|\rho'| + |\rho''| = |\rho_r|$, and get $LP \xrightarrow{\neg\mathtt{lm}*} L'P \xrightarrow{\mathtt{o}*} L'P'$. Now assuming that this is not $\prec_r$-standard leads to a contradiction to the definition at the seam, since all linear positions in L'' are $\prec_r$ with respect to those in P. □

In order to prove that also inner reductions can be standardized, we need to introduce the notion of *outer shape* of a term.

Definition 19. The *outer shape* $\ell(M)[\![\cdot]\!]$ of a term M is a context that is M with holes replacing all exponential arguments of M's bags.

Formally, extending the definition to bags, we define $\ell(\,.\,)[\![\cdot]\!]$ inductively as follows.

$$\ell(x)[\![\cdot]\!] = x, \quad \ell(\lambda x.M)[\![\cdot]\!] = \lambda x.\ell(M)[\![\cdot]\!], \qquad \ell(MP)[\![\cdot]\!] = \ell(M)[\![\cdot]\!]\ell(P)[\![\cdot]\!],$$
$$\ell(1)[\![\cdot]\!] = 1, \quad \ell([M]\cdot P)[\![\cdot]\!] = [\ell(M)[\![\cdot]\!]]\cdot\ell(P)[\![\cdot]\!], \quad \ell([M^!]\cdot P)[\![\cdot]\!] = [[\![\cdot]\!]^!]\ell(P)[\![\cdot]\!].$$

Property 20.

(i) $M \xrightarrow{\mathtt{i}*} N$ if and only if $\ell(M)[\![\cdot]\!] = \ell(N)[\![\cdot]\!]$, and there are k terms M_i' and k terms N_i' such that $M = \ell(M)_a[\![\vec{M}_i']\!]$, $N = \ell(M)_a[\![\vec{N}_i']\!]$ and $M_i' \xrightarrow{\mathtt{nd}*} N_i'$ for each i.

(ii) If $M = \ell(M)_a[\![\vec{M}_i']\!]$ and $\rho_i : M_i' \xrightarrow{\mathtt{nd}*} M_i''$ are standard, then there is a standard $\rho' : M \xrightarrow{\mathtt{i}*} \ell(M)_a[\![\vec{M}_i'']\!]$.

Proof.

i) The if direction is a direct consequence of how `i` is defined and of context closedness of the reduction. We thus move to the only if direction.

First, let us show that the property to prove is preserved by composition of reduction chains.

Suppose $M \xrightarrow{\mathtt{i}*} N \xrightarrow{\mathtt{i}*} O$ with $M = \ell(M)[\vec{M}_i']$, $N = \ell(M)_{a_1}[\vec{N}_i'] = \ell(M)_{a_2}[\vec{N}_i'']$ and $O = \ell(M)_{a_3}[\vec{O}_i']$. We can suppose $a_1 = a_2$ by re-indexing (namely using $\ell(N)_{a_1}[\vec{N}''_{a_2^{-1}(a_1(i))}]$ and $\ell(O)_{a_3'}[\vec{N}''_{a_2^{-1}(a_1(i))}]$ with $a_3' = a_3 \circ a_2^{-1} \circ a_1$). So we just forget the bijections employed, and then we have by hypothesis $M_i' \xrightarrow{\mathtt{nd}*} N_i' = N_i'' \xrightarrow{\mathtt{nd}*} O_i'$, which is what is needed.

Now, we can prove the property by reducing to the case of a single inner reduction, as composing multiple ones of them preserves the property.

Take $M \xrightarrow{\mathtt{i}} N$: the result follows by a straightforward induction on how the reduction is defined.

ii) The idea is that the reductions in the subterms can be freely rearrenged.

Let us reason by generalizing to expressions and by structural induction on $\mathbb{A}$.

Case 1. $\mathbb{A} = x$ or $\mathbb{A} = 1$: nothing to prove.

Case 2. $\mathbb{A} = \lambda x.N$: straightforward application of inductive hypothesis.

Case 3. $\mathbb{A} = NP$, with $\ell(A)[\![\cdot]\!] = \ell(N)[\![\cdot]\!]\ell(P)[\![\cdot]\!]$: we can partition M_i' into what goes in $\ell(N)[\![\cdot]\!]$ and what goes in $\ell(P)[\![\cdot]\!]$. We can suppose that $\mathbb{A} = (\ell(N)[M_1',\ldots,M_h'])(\ell(P)[M_{h+1}',\ldots,M_k'])$ without loss of generality, and by inductive hypothesis get standard $\sigma : N \xrightarrow{\mathtt{i}*} N'$ and $\rho : P \xrightarrow{\mathtt{i}*} P'$ (with N' and P' the correct pluggings of $\ell(N)$ and $\ell(P)$).

Now, if we reduce $\mathbb{A} = NP \xrightarrow{\mathtt{i}*} N'P \xrightarrow{\mathtt{i}*} N'P'$ following first σ and then ρ, the resulting reduction must be standard as all positions in P are greater than those in N according to $\prec_r$.

Case 4. $\mathbb{A} = [N]\cdot P$: exactly as above, but without any constraint on the order in which the reductions are composed.

Case 5. $\mathbb{A} = [N^!] \cdot P$, with $\ell(\mathbb{A}) = [[\![\cdot]\!]^!] \cdot \ell(P)[\![\cdot]\!]$: suppose that $M'_1 = N$ and $P = \ell(P)[M'_2, \cdots, M'_k]$. By inductive hypothesis we have a standard $\rho : P \xrightarrow{\mathtt{i}*} P' = \ell(P)[M''_i]_{i=2}^k$, and as positions in $[N^!]$ and *non-linear* positions in P are incomparable, we can freely combine the reductions on M'_1 and P to get a standard one. □

Now we are able to show the desired result.

Theorem 21 (Standardization)**.** *If $M \overset{\mathtt{nd}*}{\to} M'$, then there is a $\prec_r$-standard chain from M to M'.*

Proof. By structural induction on M', the term where the reduction ends. First, applying Theorem 8, we get $\sigma : M \xrightarrow{\mathtt{o}*} M''$ and $\rho : M'' \xrightarrow{\mathtt{ndi}*} M'$. Now we strive to obtain two standard chains $\sigma' : M \xrightarrow{\mathtt{o}*} M''$ and $\rho' : M'' \xrightarrow{\mathtt{ndi}*} M'$ to obtain the chain $\sigma'\rho'$ which is standard by Lemma 17. The existence of a standard σ' is assured directly by Lemma 18, so we need to concentrate on finding ρ'. By using Property 20(i), we get $M'' = \ell(M')[\![N_1, \ldots, N_k]\!]$, $M' = \ell(M')[\![N'_1, \ldots, N'_k]\!]$ and $\rho_i : N_i \xrightarrow{\mathtt{nd}*} N'_i$. As all N'_i are structurally strictly smaller than M', we can apply inductive hypothesis on each ρ_i and get standard $\rho'_i : N_i \xrightarrow{\mathtt{nd}*} N'_i$. Then using Property 20(ii) we can glue back those reductions into the standard reduction $\rho' : M'' \xrightarrow{\mathtt{i}*} M'$. □

Example 22. Let $I = \lambda x.x$, $M_1 = I[((\lambda xy.x)[I^!][I^!])^!]$, $M_2 = I[I^!]$, and let $M = \lambda x.x[M_1{}^!, M_2{}^!]$. The following reduction is standard: $M_1 = I[((\lambda xy.x)[I^!][I^!])^!] \overset{\mathtt{lm}}{\to} (\lambda xy.x)[I^!][I^!] \overset{\mathtt{lm}}{\to} (\lambda y.I)[I^!] \overset{\mathtt{lm}}{\to} I$. As $M_2 \overset{\mathtt{lm}}{\to} I$, the following is standard too

$$\lambda x.x[(I[((\lambda xy.x)[I^!][I^!])^!])^!, (I[I^!])^!] \overset{\mathtt{i}}{\to} \lambda x.x[((\lambda xy.x)[I^!][I^!])^!, (I[I^!])^!] \overset{\mathtt{i}}{\to}$$
$$\lambda x.x[((\lambda xy.x)[I^!][I^!])^!, I^!] \overset{\mathtt{i}}{\to} \lambda x.x[((\lambda y.I)[I^!])^!, I^!] \overset{\mathtt{i}}{\to} \lambda x.x[I^!, I^!].$$

Let us notice that, as opposed to the weak form of standardization given in Theorem 8, the $\prec_r$-standardization does not hold for parallel reduction. A counterexample is the following.

Example 23. Let I_0 and I_1 denote two occurrences of the identity $\lambda x.x$, and let $M = I_0[I_1[x^!, y^!]] \overset{\mathtt{g}}{\to} I_0[x] + I_0[y] \overset{\mathtt{g}}{\to} x + I_0[y]$ by reducing the inner redex first. But reducing the leftmost redex first we obtain $M \overset{\mathtt{g}}{\to} I_1[x^!, y^!] \overset{\mathtt{g}}{\to} x + y$. So the previous result cannot be obtained by a standard reduction.

4 Solvability Machine

The standardization result proved in the previous section allows us to design an abstract reduction machine characterizing the may-solvable terms in Λ^r. A term of λ-calculus is solvable whenever there is a outer-context reducing it to the identity [1]. In the resource calculus, terms appear in formal sums, so (at least) two different notions of solvability arise, related to a may and must operational semantics, respectively. We will treat the former only.

Definition 24. A simple term M is **may-solvable** whenever there is a linear applicative–context $C[\![\cdot]\!]$ such that $C[\![M]\!] \xrightarrow{\mathtt{nd}*} I$.

May-solvability has been completely characterized from both a syntactical and logical point of view in [8]. Syntactically, a term M is may-solvable if and only if it is may-outer normalizable. An expression is an **outer normal form** (onf) if it has no redex but under the scope of a $(\,)^!$, and consequently a term M is **may-outer normalizable** if and only if $M \overset{\mathtt{nd}*}{\to} N$, where N is a onf (N is called a $monf$ of M). Logically,

$$\frac{M \Downarrow_{nd} M'}{\lambda x.M \Downarrow_{nd} \lambda x.M'}\ (\lambda) \qquad \frac{M \text{ is in onf}}{M \Downarrow_{nd} M}\ (end) \qquad \frac{P_i \Downarrow_b P_i' \quad (1 \le i \le m)}{xP_1...P_m \Downarrow_{nd} xP_1'...P_m'}\ (head)$$

$$\frac{M\{0/x\}P_1...P_m \Downarrow_{nd} M'}{(\lambda x.M)1P_1...P_m \Downarrow_{nd} M'}\ (0) \qquad \frac{M\langle N/x\rangle = M' + \mathbb{A} \quad (\lambda x.M')PP_1...P_m \Downarrow_{nd} M''}{(\lambda x.M)[N]\cdot PP_1...P_m \Downarrow_{nd} M''}\ (\beta)$$

$$\frac{M\{N+x/x\} = M' + \mathbb{A} \quad (\lambda x.M')PP_1...P_m \Downarrow_{nd} M''}{(\lambda x.M)[N^!]\cdot PP_1...P_m \Downarrow_{nd} M''}\ (!\beta)$$

(a) The ND reduction machine.

$$\frac{}{1 \Downarrow_b 1}\ (1_b) \qquad \frac{M \Downarrow_{nd} N \quad P \Downarrow_b P'}{[M]\cdot P \Downarrow_b [N]\cdot P'}\ (b) \qquad \frac{P \Downarrow_b P'}{[M^!]\cdot P \Downarrow_b [M^!]\cdot P'}\ (!b)$$

(b) The auxiliary *B* machine

a particular intersection type assignment system has been defined, typing all and only the may-solvable terms.

We now will complete the job, characterizing may solvability from an operational point of view. The following property is obvious.

Property 25. M is in onf if and only if $\mathscr{L}(M) = \emptyset$.

The abstract reduction machine (called ND-machine) proves statements of the shape $M \Downarrow_{nd} N$, where M, N are simple terms and N is a onf. The ND-machine uses an auxiliary machine, the B-machine, performing the reductions on bags. The two machines are shown in Figure 4.

Some comments are in order. First of all, the machine performs the baby outer reduction, on a leftmost redex. Rules (λ), (end) and (0) are self-explanatory. Rule $(head)$ implements the definition of $monf$; note that in this rule the order in which the arguments are reduced does not matter. Non-determinism appears in rules (β) and $(!\beta)$. Indeed, if the result of the substitution is a sum, one of its addends is randomly chosen. The auxiliary machine B performs the reductions on bags. Note that the rule $(!b)$ implements the notion of outer-reduction. Remember that 0 is not a term, so it can be neither an input nor an output of the machine. So in rules (0), (β) and $(!\beta)$ the machine transition is undefined if the result of the substitution is 0. We will write $M \Uparrow_{nd}$ to denote that for any run of the machine on M either it does not stop or it is undefined.

Example 26. $(\lambda zy.y)[x] \Uparrow_{nd}$. In fact, trying to apply rule β, the machine needs to compute $(\lambda y.y)\langle x/z\rangle$, which is equal to 0, so the premises of the rule are not satisfied.
$(\lambda x.x[x^!])(\lambda x.x[x^!]) \Uparrow_{nd}$. In fact, the machine on this input does not stop. Notice that this term corresponds to an unsolvable term in the λ-calculus.
Let $F = \lambda xy.y$. Then $(\lambda x.y[x][x])[F,I][lm]$ reduces non deterministically to $y[F][I] + y[I][F]$. It is easy to check that there are two machine computations such that in one $(\lambda x.y[x][x])[F,I] \Downarrow_{nd} y[F][I]$ while in the other $(\lambda x.y[x][x])[F,I] \Downarrow_{nd} y[I][F]$.
$(\lambda x.y[x^!])[I^!,F^!] \xrightarrow{\mathtt{g}} y[I^!,F^!]$, by reducing the leftmost redex. The unique machine computation for this input gives $(\lambda x.y[x^!])[I^!,F^!] \Downarrow_{nd} y[I^!,F^!]$.

Theorem 27.

(i) *(Soundness) If $M \Downarrow_{nd} N$ then $M \xrightarrow{\mathtt{lm}*} N$, and N is a* onf.

(ii) *(Completeness) Let M be may-outer-normalizable and let N be a* monf *of M. There is a machine's computation proving $M \Downarrow_{nd} N'$, where N' is a* monf *of M and $N' \xrightarrow{\neg\mathtt{lm}*} N$.*

Proof (sketch). Point (i) is proved by mutual induction on the rules of the two machines. Point (ii) is an immediate consequence of the $\prec_r$-standardization property. □

Acknowledgements. We would like to thank Michele Pagani for his interesting and useful suggestions.

References

[1] Henk Barendregt (1984): *The Lambda-Calculus, its Syntax and Semantics*. Stud. Logic Found. Math., vol. 103, North-Holland.

[2] Gérard Boudol (1993): *The Lambda-Calculus with Multiplicities*. *INRIA Report 2025* Available at `citeseer.ist.psu.edu/article/boudol93lambdacalculus.html`.

[3] Ugo de'Liguoro & Adolfo Piperno (1995): *Non Deterministic Extensions of Untyped Lambda-Calculus*. *Inf. Comput.* 122(2), pp. 149–177, doi:10.1006/inco.1995.1145.

[4] Thomas Ehrhard & Laurent Regnier (2003): *The Differential Lambda-Calculus*. *Theor. Comput. Sci.* 309(1), pp. 1–41, doi:10.1016/S0304-3975(03)00392-X.

[5] Thomas Ehrhard & Laurent Regnier (2006): *Böhm Trees, Krivine's Machine and the Taylor Expansion of Lambda-Terms*. In: *CiE*, *LNCS* 3988, pp. 186–197, doi:10.1007/11780342_20.

[6] Thomas Ehrhard & Laurent Regnier (2008): *Uniformity and the Taylor Expansion of Ordinary Lambda-Terms*. *Theor. Comput. Sci.* 403(2-3), pp. 347–372, doi:10.1016/j.tcs.2008.06.001.

[7] Michele Pagani & Simona Ronchi Della Rocca (2010): *Linearity, Non-determinism and Solvability*. *Fundamenta Informaticae* 104, pp. 1–30, doi:10.3233/FI-2010-324.

[8] Michele Pagani & Simona Ronchi Della Rocca (2010): *Solvability in Resource Lambda-Calculus*. In Luke Ong, editor: *FOSSACS, Lecture Notes in Comp. Sci.* 6014, pp. 358–373, doi:10.1007/978-3-642-12032-9_25.

[9] Michele Pagani & Paolo Tranquilli (2009): *Parallel Reduction in Resource Lambda-Calculus*. In: *APLAS*, *LNCS* 5904, pp. 226–242, doi:10.1007/978-3-642-10672-9_17.

[10] Paolo Tranquilli (2009): *Nets between Determinism and Nondeterminism*. Ph.D. thesis, Università Roma Tre/Université Paris Diderot (Paris 7).

[11] Paolo Tranquilli (2011): *Intuitionistic differential nets and lambda-calculus*. *Theor. Comput. Sci.* 412(20), pp. 1979–1997, doi:10.1016/j.tcs.2010.12.022.