# Pattern Avoidability with Involution

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An infinte word $w$ avoids a pattern $p$ with the involution $\theta$ if there is no substitution for the variables in $p$ and no involution $\theta$ such that the resulting word is a factor of $w$. We investigate the avoidance of patterns with respect to the size of the alphabet. For example, it is shown that the pattern $\alpha\,\theta(\alpha)\,\alpha$ can be avoided over three letters but not two letters, whereas it is well known that $\alpha\,\alpha\,\alpha$ is avoidable over two letters.

## 1 Introduction

The avoidability of patterns in infinite words is an old area of interest with a first systematic study going back to Thue [5, 6]. This field includes rediscoveries and studies by many authors over the last one hundred years; see for example [2] and [1] for surveys. In this article, we are concerned with a variation of the theme by considering avoidable patterns with involution. An involution $\theta$ is a mapping such that $\theta^2$ is the identity. We consider morphic, where $\theta(uv) = \theta(u)\theta(v)$, and antimorphic involutions, where $\theta(uv) = \theta(v)\theta(u)$. The subject of this article draws quite some motivation from applications in biology where the Watson-Crick complement corresponds to an antimorphic involution in our case. Our considerations are more general, however, by considering any alphabet size and also morphic involutions.

During the review phase of this article, James Currie [3] presented a solution for all those patterns under involution in $\{\alpha, \theta(\alpha)\}^*$ that we do not consider here, which leads to a characterization of the avoidance index for all unary patterns under involution.

## 2 Preliminaries

Our notation is guided by what is commonly found in the literature, see for example the first chapter of [4] as a reference. Let $\Sigma$ be a finite alphabet of *letters* and $\Sigma^*$ denote all *finite* and $\Sigma^\omega$ denote all (right-) *infinite* words over $\Sigma$. Let $\varepsilon$ denote the empty word. Letters are usually denoted by $a$, $b$, or $c$, and words over $\Sigma$ are usually denoted by $u$, $v$, or $w$ in this paper. The $i$-th letter of a word $w$ is denoted by $w_{[i]}$, that is, $w = w_{[1]}w_{[2]}\cdots w_{[n]}$ if $w$ is finite, and the length $n$ of $w$ is denoted by $|w|$ as usual.

Besides $\Sigma$ we need another finite set $E$ of symbols. The elements of $E$ are called *variables* and we usually denote them by $\alpha$, $\beta$, or $\gamma$. Words in $E^*$ are called *patterns*. For example $\alpha\beta\alpha \in E^*$ is a pattern consisting of the variables $\alpha$ and $\beta$ in $E$. We assign to every pattern a *pattern language* over the alphabet $\Sigma$. This language contains every word, that can be generated by substituting all variables in the pattern by non-empty words in $\Sigma^*$. For example the pattern language of the pattern $\alpha\alpha$ over $\Sigma = \{a, b\}$ is $\{aa, bb, aaaa, abab, baba, bbbb, \dots\}$.

We say that a word $w$ *avoids* a pattern, if no factor of $w$ exists, that is in the pattern language. On the other hand, if a factor of $w$ is an element of the pattern language, we say $w$ *contains* the pattern. If for a

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given pattern $e$ and an alphabet $\Sigma$ with $k$ elements a word $w \in \Sigma^{\omega}$ exists that avoids $e$, then we say that $e$ is *$k$-avoidable*. Otherwise we call $e$ *$k$-unavoidable*. We call $k \in \mathbb{N}$ the *avoidance index* $\mathscr{V}(e)$ of a pattern $e \in E^*$, if $e$ is $k$-avoidable and $k$ is minimal. If no such $k$ exists, we define $\mathscr{V}(e) = \infty$.

Let $f\colon \{a,b\}^* \to \{a,b\}^*$ with $a \mapsto ab$ and $b \mapsto ba$. The fixpoint $t = \lim_{k\to\infty} f^k(a)$ exists and is called *Thue–Morse word*. The following result is a classical one.

**Theorem 1** ([5, 6])**.** *The Thue–Morse word avoids the patterns $\alpha\alpha\alpha$ and $\alpha\beta\alpha\beta\alpha$.*

## 3 Patterns with Involution

For introducing patterns with involution, we extend the set of pattern variables $E$ by adding $\theta(\alpha)$ for all variables $\alpha \in E$ and some involution $\theta$. For the rest of the article, we will stick to this definition of $E$. Given a morphic or antimorphic involution, we build the corresponding pattern language by replacing the variables by non-empty words and, for variables of the form $\theta(\alpha)$, by applying the involution after the substitution.

For example, let $\theta$ be the morphic involution with $a \mapsto b$ and $b \mapsto a$ over $\Sigma = \{a,b\}$, and let the pattern be $\alpha\,\theta(\alpha)$. We get the pattern language $\{ab, ba, aabb, abba, baab, bbaa, \dots\}$. Every word in $\{a,b\}^{\omega} \setminus (a^{\omega} \cup b^{\omega})$ contains the pattern $\alpha\,\theta(\alpha)$ for the morphic involution $\theta$ with $a \mapsto b$ and $b \mapsto a$.

**Observation 2.** *Let $\theta$ be a morphic or antimorphic involution and not the identity or reversal mapping. Then every pattern, that contains variables of the $\alpha$ and $\theta(\alpha)$, is avoidable.*

Indeed, since $\theta$ is not the identity or reversal mapping, a letter $a \in \Sigma$ with $\theta(a) \neq a$ exists. Therefore $w = a^{\omega}$ avoids every pattern that includes variables $\alpha$ and $\theta(\alpha)$.

Because of this observation we do not have to examine, if patterns are avoidable or unavoidable for a given involution. So we now change the point of view. For a given pattern $e \in E^*$, we either look at all morphic or all antimorphic involutions $\Sigma^* \to \Sigma^*$ at the same time. So, we examine, for example, if an infinite word $w \in \Sigma^{\omega}$ exists, that avoids a pattern $e$ for all morphic involutions.

**Definition 3.** *Let $e \in E^*$ be a pattern, possibly with variables of the form $\theta(\alpha)$. We call $k \in \mathbb{N}$ the morphic (antimorphic) $\theta$-avoidance index $\mathscr{V}_{\mathrm{m}}^{\theta}(e)$ ($\mathscr{V}_{\mathrm{a}}^{\theta}(e)$) of $e \in E^*$, if an infinite word $w \in \Sigma^{\omega}$ over $\Sigma$ with $|\Sigma| = k$ exists, that avoids the pattern $e$ for all morphic (antimorphic) involutions $\Sigma^* \to \Sigma^*$ and $k$ is minimal. If this doesn't hold for any $k \in \mathbb{N}$, we define $\mathscr{V}_{\mathrm{m}}^{\theta}(e) = \infty$ ($\mathscr{V}_{\mathrm{a}}^{\theta}(e) = \infty$).*

We establish the first facts about avoidance of pattern $\alpha\,\theta(\alpha)\,\alpha$.

**Lemma 4.** *Let $\Sigma$ be a binary alphabet. Then there is no word $w \in \Sigma^{\omega}$, that avoids the pattern $\alpha\,\theta(\alpha)\,\alpha$ for all morphic involutions $\theta\colon \Sigma^* \to \Sigma^*$. That is, $\mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\,\theta(\alpha)\,\alpha) > 2$.*

*Proof.* Let $\Sigma = \{a,b\}$. We try to construct a word $w \in \Sigma^{\omega}$, that avoids $e = \alpha\,\theta(\alpha)\,\alpha$ for all morphic involutions and bring this to a contradiction. For example, this word must not contain *aaa*, *bbb*, *aba* or *bab* as a factor. Without loss of generality $w$ begins with $a$.

Case 1: Assumed the word $w$ begins with $ab$. Then this prefix must be followed by $b$, $abb <_{\mathrm{p}} w$. The next letter must be an $a$, the fifth must be an $a$ too. So we have $abbaa <_{\mathrm{p}} w$. If the following letter is an $a$, $aaa$ is a factor of $w$. So the next letter must be the letter $b$. But for the morphic involution $\theta$ with $a \mapsto b$ and $b \mapsto a$ the word $ab\theta(ab)ab$ is a factor of $w$.

Case 2: The argument for the case $aa \leq_{\mathrm{p}} w$ is analogous to case 1. □

The proof of the following lemma is analogous to the previous one.

**Lemma 5.** *Let $\Sigma$ be a binary alphabet. There is no word $w \in \Sigma^{\omega}$, that avoids the pattern $\alpha\,\theta(\alpha)\,\alpha$ for all antimorphic involutions $\theta\colon \Sigma^* \to \Sigma^*$. That is, $\mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\,\theta(\alpha)\,\alpha) > 2$.*

## 4 Main Result

In this section, we establish the $\theta$-avoidance indices for the pattern $\alpha\,\theta(\alpha)\,\alpha$ in the morphic and antimorphic case. We start with the morphic case.

**Theorem 6.** *It holds that* $\mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\theta(\alpha)\alpha) = 3$.

*Proof.* Let $\Sigma$ an alphabet with three elements, $\Sigma = \{a,b,c\}$. Let $v$ be the infinitely long Thue–Morse word over the letters $a'$ and $b'$. Furthermore let $w \in \Sigma^{\omega}$ be the word, that is the outcome of replacing every $a'$ in $v$ by $aacb$ and $b'$ by $accb$. We will show, that $w$ avoids the pattern $\alpha\theta(\alpha)\alpha$ for all morphic involutions. For better readability, we define $x = aacb$ and $y = accb$.

We assume it exists a morphic involution $\theta$ and a substitution for $\alpha$, such that $\alpha\theta(\alpha)\alpha$ is a factor of $w$. Proof by contradiction. First, we examine the possibilities of replacing the variable $\alpha$ by words $u \in \Sigma^{+}$ of length $|u| < 7$. The word $u\,\theta(u)\,u$ has a maximal length of 18. Therefore there must exist a morphic involution so that $u\,\theta(u)\,u$ is a factor of a word $w' \in \{x,y\}^{6}$. Because of Theorem 1, the words $xxx$, $yyy$, $xyxyx$ and $yxyxy$ can not be a factor of $w'$. A computer program can easily check these finite possibilities with the result, that no words $u$ and $w'$ exist, which fulfill the conditions. Now we assume $\alpha$ gets replaced by a word $u \in \Sigma^{+}$ with $|u| \geq 7$. Then, the word $u$ contains $aacb$ or $accb$. Without loss of generality, $u$ contains $aacb$. Therefore, $\theta(u)$ contains the factor $\theta(aac) = \theta(a)\,\theta(a)\,\theta(c)$. In addition $\theta(u)$ and for this reason $\theta(a)\,\theta(a)\,\theta(c)$ is a factor of $w$. There are only two possibilities for two succeeding identical letters in $w$. Either these letters are two letters $c$ followed by the letter $b$, or two letters $a$ are followed by the letter $c$. This implies, that $u\,\theta(u)\,u$ can only be a factor of $w$, if $\theta$ is the identity mapping. Furthermore this implies $|u| = 4\cdot k$ for a $k \in \mathbb{N}$. This is visualized in Fig. 1, where $w_i, w_{i'}, w_{i''} \in \{x,y\}$ holds for all $0 \leq i \leq k$. If the word $(w_0)_{[2]}(w_0)_{[3]}(w_0)_{[4]}$ or $(w_0)_{[1]}(w_0)_{[2]}(w_0)_{[3]}(w_0)_{[4]} = w_0$ is a prefix of the first $u$ in Fig. 1, then the following equations apply:

$$\begin{array}{ccccc} w_0 & = & w_{0'} & = & w_{0''} \\ w_1 & = & w_{1'} & = & w_{1''} \\ \vdots & & \vdots & & \vdots \\ w_{k-1} & = & w_{k-1'} & = & w_{k-1''} \end{array}$$

The word $w_0w_1\dots w_{k-1}\,w_{0'}w_{1'}\dots w_{k-1'}\,w_{0''}w_{1''}\dots w_{k-1''} = (w_0w_1\dots w_{k-1})^3$ is a factor of $w$. Because of $w_i \in \{x,y\}$ for all $0 \leq i \leq k-1$, this is a contradiction to Lemma 1. On the other hand, if only $(w_0)_{[3]}(w_0)_{[4]}$ or $(w_0)_{[4]}$ is a prefix of $u$, then $w_0 \neq w_{0'}$ is possible. But in this case $(w_{k''})_{[1]}(w_{k''})_{[2]}$ or $(w_{k''})_{[1]}(w_{k''})_{[2]}(w_{k''})_{[3]}$ is a suffix of the third $u$. This implies

$$\begin{array}{ccccc} w_1 & = & w_{1'} & = & w_{1''} \\ w_2 & = & w_{2'} & = & w_{2''} \\ \vdots & & \vdots & & \vdots \\ w_{k} & = & w_{k'} & = & w_{k''} \end{array}$$

and $w_1w_2\dots w_k\,w_{1'}w_{2'}\dots w_{k'}\,w_{1''}w_{2''}\dots w_{k''} = (w_1w_2\dots w_k)^3$ is a factor of $w$. Again, this is a contradiction to Lemma 1. The theorem follows with Lemma 4. ☐

The result of Theorem 6 transfers also to the antimorphic case.

**Theorem 7.** *It holds that* $\mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\theta(\alpha)\alpha) = 3$.

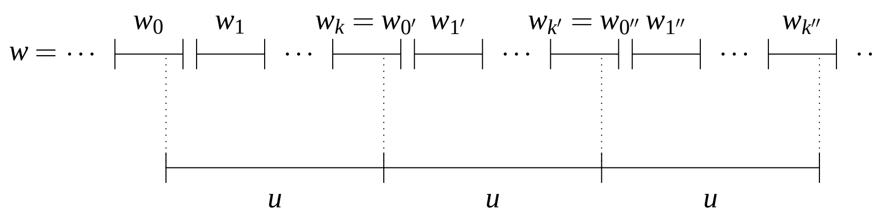


Figure 1: Part of $w$ to illustrate the factor $uuu$

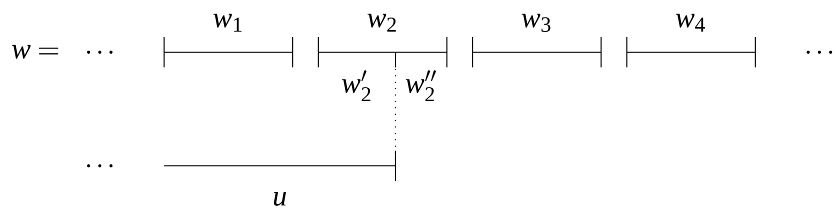


Figure 2: Part of $w$ and the factor $u$ of $w$

*Proof.* This proof follows the proof of the previous theorem. Let $\Sigma$ be an alphabet with three elements, $\Sigma = \{a, b, c\}$. Further, let $v$ be the Thue-Morse word over the letters $a'$ and $b'$. Let $w \in \Sigma^\omega$ be the word, that we get by replacing $a'$ in $v$ by $aabbc$ and $b'$ by $aaccb$. We will show, that $w$ avoids the pattern $\alpha\,\theta(\alpha)\,\alpha$ for all antimorphic involutions. For better readability, we define $x = aabbc$ and $y = aaccb$.

We assume that there exists an antimorphic involution and a substitution of $\alpha$ by a word $u \in \Sigma^+$ in such a way, that $u\,\theta(u)\,u$ is a factor of $w$. First we suppose that $|u| < 9$ holds. The word $u\,\theta(u)\,u$ then has a maximal length of 24 and $u\,\theta(u)\,u$ is factor of a word $w' \in \{x, y\}^6$. The word $xxx$, $yyy$, $xyxyx$, and $yxyxy$ must not be a factor of $w'$ because of Lemma 1. A computer program can check these finite possibilities with the result, that no words $u$ and $w'$ exist that fulfill these conditions for an antimorphic involution $\theta$. So ,$|u| \geq 9$ must hold and $u$ contains at least one word $x$ or $y$ completely. We now look at the first $u$ of the factor $u\,\theta(u)\,u$ of $w$. Let $w_1 w_2' \leq_{\mathrm{s}} u$ with $w_1, w_2 \in \{x, y\}$, $w_2 = w_2' w_2''$ and $|w_2'| < 5$. We get Fig. 2 where $w_3, w_4 \in \{x, y\}$. Without loss of generality, let $w_1 = x = aabbc$. Then $\theta(u)$ and therefore $w_2 w_3 w_4$ contains the word $\theta(aabbc) = \theta(c)\,\theta(b)\,\theta(b)\,\theta(a)\,\theta(a)$ with length 5 as a factor. Hence we look at the following words:

$$
\begin{aligned}
xx &= aabbc\,aabbc \\
xy &= aabbc\,aaccb \\
yx &= aaccb\,aabbc \\
yy &= aaccb\,aaccb\ .
\end{aligned}
$$

Only $xx$ contains $\theta(c)\,\theta(b)\,\theta(b)\,\theta(a)\,\theta(a)$ for the antimorphic involution $\theta$ with $a \mapsto b$, $b \mapsto a$, and $c \mapsto c$. Because of $w_1 = x$, the equation $w_2 w_3 = xx$ is a contradiction to Lemma 1. The case $w_2 w_3 w_4 = yxx$ remains. Now there are five possibilities for the position of $u$, see Fig. 3. It is easy to check, that in all five cases $\theta(u) \leq_{\mathrm{p}} w_2'' w_3 w_4$ respectively $w_2'' w_3 w_4 \leq_{\mathrm{p}} \theta(u)$ doesn't hold. So our assumption, that there exists an antimorphic involution $\theta$ and a word $u \in \Sigma^+$ with $u\,\theta(u)\,u$ is a factor of $w$, was wrong. The theorem follows with Lemma 5. $\square$

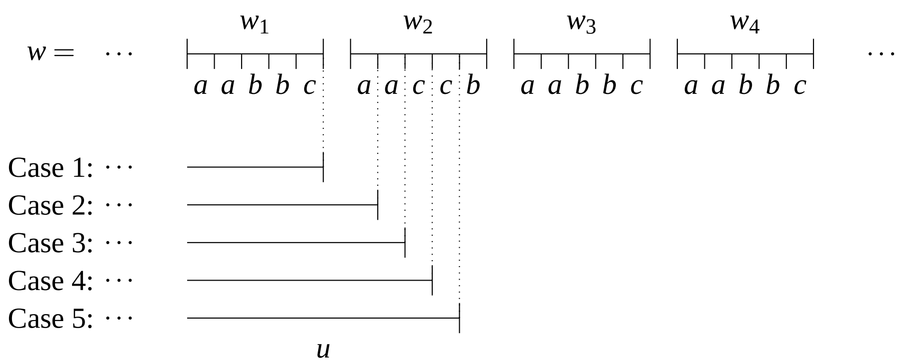


Figure 3: Illustration of possible positions of the factor $u$ of $w$

## 5 Complementary Patterns

In this section, patterns similar to $\alpha\,\theta(\alpha)\,\alpha$ are considered.

For the next lemma we need a further definition. Let $e \in E^*$ be a pattern consisting of variables of the form $\alpha$ and $\theta(\alpha)$ and $e'$ be the pattern that we get, when all variables $\alpha$ and $\theta(\alpha)$ in $e$ are switched. We call $e' \in E$ the $\theta$-*complementary* pattern of $e$. For example the $\theta$-complementary pattern of $\alpha\,\alpha\,\theta(\alpha)\,\beta$ is $\theta(\alpha)\,\theta(\alpha)\,\alpha\,\theta(\beta)$. For this definition it doesn't matter if morphic or antimorphic involutions are examined.

**Lemma 8.** *Let $e \in E^*$ be a pattern and $e' \in E$ be the $\theta$-complementary pattern of $e$. Then $\mathscr{V}_{\mathrm{a}}^{\theta}(e) = \mathscr{V}_{\mathrm{a}}^{\theta}(e')$ and $\mathscr{V}_{\mathrm{m}}^{\theta}(e) = \mathscr{V}_{\mathrm{m}}^{\theta}(e')$.*

*Proof.* First of all we show $\mathscr{V}_{\mathrm{m}}^{\theta}(e) = \mathscr{V}_{\mathrm{m}}^{\theta}(e')$. For better readability, we replace the variable $\alpha$ in the pattern $e'$ by $\alpha'$ and $\theta(\alpha)$ by $\theta(\alpha')$. We assume a word $w \in \Sigma^{\omega}$ contains the pattern $e$ for a morphic involution and a substitution of $\alpha$ by $u \in \Sigma^+$. Then $w$ contains the pattern $e'$ for the same morphic involution by substituting $\alpha'$ by $\theta(u)$. Symmetry reasons imply:

$$\begin{aligned} &\text{It exists a morphic involution } \theta \text{ so that } w \text{ contains the pattern } e. \\ \Leftrightarrow\ &\text{It exists a morphic involution } \theta' \text{ so that } w \text{ contains the pattern } e'. \end{aligned}$$

By negation we get:

$$\begin{aligned} &\text{The word } w \in \Sigma^{\omega} \text{ avoids the pattern } e. \\ \Leftrightarrow\ &\text{The word } w \in \Sigma^{\omega} \text{ avoids the pattern } e'. \end{aligned}$$

The equation $\mathscr{V}_{\mathrm{m}}^{\theta}(e) = \mathscr{V}_{\mathrm{m}}^{\theta}(e')$ follows. The proof of $\mathscr{V}_{\mathrm{a}}^{\theta}(e) = \mathscr{V}_{\mathrm{a}}^{\theta}(e')$ is identical. □

Note the following $\theta$-free patterns; see [1].

**Observation 9.** *The patterns $\alpha\alpha$, $\alpha\alpha\beta$, $\beta\alpha\alpha$, $\alpha\alpha\beta\alpha$, $\alpha\beta\beta\alpha$, $\alpha\alpha\beta\beta$, $\alpha\beta\alpha\beta$, $\alpha\alpha\beta\alpha\alpha$, and $\alpha\alpha\beta\alpha\beta$ are* 2*-unavoidable and* 3*-avoidable.*

**Lemma 10.** *Let $e \in E^*$ be a pattern, that contains the variables $\alpha$ and $\theta(\alpha)$. Further, $e$ contains no other variable of the form $\theta(\gamma)$. Let $e'$ be the pattern when all occurrences of $\theta(\alpha)$ in $e$ are replaced by $\alpha$. The pattern $e''$ obtained when all occurrences of $\theta(\alpha)$ in $e$ are replaced by a new variable $\beta$.*

*Then $\mathscr{V}(e') \le \mathscr{V}_{\mathrm{m}}^{\theta}(e) \le \mathscr{V}(e'')$ and $\mathscr{V}_{\mathrm{a}}^{\theta}(e) \le \mathscr{V}(e'')$.*

*Proof.* The relation $\mathscr{V}(e') \leq \mathscr{V}_{\mathrm{m}}^{\theta}(e)$ holds, since the morphic $\theta$-avoidance index considers all morphic involutions, including the identity mapping. Now say $\mathscr{V}(e'') = k$, i.e., a word $w \in \Sigma^{\omega}$ exists, that avoids the pattern $e''$.Then this word also avoids the pattern $e$ for all morphic and antimorphic involutions. Therefore the relations $\mathscr{V}_{\mathrm{m}}^{\theta}(e) \leq \mathscr{V}(e'')$ and $\mathscr{V}_{\mathrm{a}}^{\theta}(e) \leq \mathscr{V}(e'')$ hold. □

**Lemma 11.** *It holds that* $\mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) = \mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) = 3$.

*Proof.* According to Observation 9 the equation $\mathscr{V}(\alpha\,\alpha\,\beta) = 3$ holds. Lemma 10 implies $\mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\,\alpha\,\theta(\alpha))$, $\mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) \leq 3$. We show by contradiction, that it holds that $\mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) \neq 2$. The proof for the relation $\mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) \neq 2$ is analogous. Assuming a word $w \in \Sigma^{\omega}$ with $\Sigma = \{a, b\}$ exists that avoids the pattern $\alpha\,\alpha\,\theta(\alpha)$ for all antimorphic involutions. Then $w$ contains neither $aa$ nor $bb$ as a factor. Without loss of generality $w$ begins with the letter $a$. It follows that $w = (ab)^{\omega}$. But $w = (ab)^{\omega}$ contains the pattern $\alpha\,\alpha\,\theta(\alpha)$ for $\alpha = ab$ and the antimorphic involution defined by $a \mapsto b$ and $b \mapsto a$. This is a contradiction to our assumption. Therefore $\mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) \neq 2$ holds and analogously $\mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) \neq 2$. We get $\mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) = \mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\,\alpha\,\theta(\alpha)) = 3$. □

**Lemma 12.** *It holds that* $\mathscr{V}_{\mathrm{a}}^{\theta}(\theta(\alpha)\,\alpha\,\alpha) = \mathscr{V}_{\mathrm{m}}^{\theta}(\theta(\alpha)\,\alpha\,\alpha) = 3$.

*Proof.* The proof is analogous to the proof of Lemma 11. □

**Corollary 13.**

1. $\mathscr{V}_{\mathrm{m}}^{\theta}(\theta(\alpha)\,\alpha\,\theta(\alpha)) = \mathscr{V}_{\mathrm{a}}^{\theta}(\theta(\alpha)\,\alpha\,\theta(\alpha)) = 3$ *by Theorem 6 and 7.*
2. $\mathscr{V}_{\mathrm{m}}^{\theta}(\theta(\alpha)\,\theta(\alpha)\,\alpha) = \mathscr{V}_{\mathrm{a}}^{\theta}(\theta(\alpha)\,\theta(\alpha)\,\alpha) = 3$ *by Lemma 11.*
3. $\mathscr{V}_{\mathrm{m}}^{\theta}(\alpha\,\theta(\alpha)\,\theta(\alpha)) = \mathscr{V}_{\mathrm{a}}^{\theta}(\alpha\,\theta(\alpha)\,\theta(\alpha)) = 3$ *by Lemma 12.*

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