The Critical Exponent is Computable for Automatic Sequences

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The critical exponent of an infinite word is defined to be the supremum of the exponent of each of its factors. For k-automatic sequences, we show that this critical exponent is always either a rational number or infinite, and its value is computable. This generalizes or recovers previous results of Krieger and others. Our technique is applicable to other situations; e.g., the computation of the optimal recurrence constant for a linearly recurrent k-automatic sequence.

1 Introduction

Let $\mathbf{a} = (a(n))_{n\geq 0}$ be an infinite sequence (or infinite word) over a finite alphabet Δ. We write $\mathbf{a}[i] = a(i)$, and we let $\mathbf{a}[i..i+n-1]$ denote the factor of length n beginning at position i.

If a finite word w is expressed in the form $x^n x'$, where $n \geq 1$ and x' is a prefix of x, then we say that w has *period* x and *exponent* $|w|/|x|$. The shortest such period is called *the* period and the largest such exponent is called *the* exponent. For example, the period of `alfalfa` is 3 and its exponent is $7/3$. The *critical exponent* of an infinite word $\mathbf{a}$ is defined to be the supremum, over all nonempty factors w of $\mathbf{a}$, of the exponent of w; it is denoted by $c(\mathbf{a})$. It is possible for the critical exponent $c(\mathbf{a})$ to be rational, irrational, or infinite. If it is rational, it is possible for $c(\mathbf{a})$ to be attained by some finite factor of $\mathbf{a}$, or not attained by any finite factor.

Critical exponents are an active subject of study. Here are just a few examples.

Example 1. Consider the Thue-Morse sequence

$$\mathbf{t} = \mathtt{0110100110010110}\cdots,$$

defined by $\mathbf{t}[i] =$ the sum, modulo 2, of the digits in the binary expansion of i. Alternatively, $\mathbf{t}$ is the fixed point, starting with $\mathbf{0}$, of the morphism μ defined by $\mathtt{0} \rightarrow \mathtt{01}$ and $\mathtt{1} \rightarrow \mathtt{10}$.

As is well-known, $\mathbf{t}$ contains no overlaps, that is, no factors of the form $axaxa$, where $a \in \{\mathtt{0},\mathtt{1}\}$ and $x \in \{\mathtt{0},\mathtt{1}\}^*$. On the other hand, $\mathbf{t}$ contains square factors such as $\mathtt{00}$. It follows that the critical exponent of $\mathbf{t}$ is 2, and this exponent is attained by a factor of $\mathbf{t}$.

Example 2. The sequence $\mathtt{0000}\cdots$ clearly has a critical exponent of ∞, as does any ultimately periodic word.

Example 3. The Rudin-Shapiro sequence $\mathbf{r} = \mathtt{00010010}\cdots$ counts the number of (possibly overlapping) occurrences of $\mathtt{11}$, modulo 2, in the base-2 expansion of n. Its critical exponent is 4 and it is attained at, for example, the factor $\mathtt{0000}$ [1].

Example 4. The sequence $\mathbf{c} = \mathtt{2102012101202102012021012102012}\cdots$, which counts the number of $\mathtt{1}$'s between consecutive occurrences of $\mathtt{0}$ in $\mathbf{t}$, is well-known to be squarefree. However, since $\mathbf{t}$ contains arbitrarily large squares — for example, the squares $\mu^n(\mathtt{00})$ — it follows that $\mathbf{c}$ contains factors of exponent arbitrarily close to 2. Thus its critical exponent is 2, but this is not attained by any finite factor.

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Example 5. Consider the Fibonacci word

$$\mathbf{f} = \mathtt{0100101001001010010100100101001001} \cdots,$$

defined to be the fixed point of the morphism $\mathtt{0} \to \mathtt{01}$ and $\mathtt{1} \to \mathtt{0}$. Then Mignosi and Pirillo [14] proved that the critical exponent of $\mathbf{f}$ is $(5+\sqrt{5})/2$, an irrational number.

Example 6. In fact, every real number greater than 1 is the critical exponent of some infinite word [12], and every real number ≥ 2 is the critical exponent of some infinite binary word [8].

Krieger [9, 10, 11] showed (among other things) that if an infinite sequence is given as the fixed point of a uniform morphism, then its critical exponent is either infinite or a rational number.

In this paper we generalize this result to the case of k-automatic sequences. An infinite sequence $\mathbf{a}$ is said to be k-*automatic* for some integer $k \geq 2$ if it is computable by a finite automaton taking as input the base-k representation of n, and having $\mathbf{a}[n]$ as the output associated with the last state encountered; see, for example, [3, 7].

For example, in Figure 1, we see an automaton generating the Thue-Morse sequence $\mathbf{t} = t_0 t_1 t_2 \cdots = \mathtt{011010011001} \cdots$. The input is n, expressed in base 2, and the output is the number contained in the state last reached.

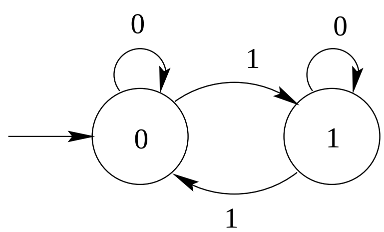

Figure 1: A finite automaton generating a sequence

As is well-known, the class of k-automatic sequences is slightly more general than the class of fixed points of uniform morphisms; the former also includes words that can be written as the image, under a coding, of fixed points of uniform morphisms [7]. An example of a word that is 2-automatic but not the fixed point of any uniform morphism is the Rudin-Shapiro sequence $\mathbf{r}$, discussed above in Example 3.

(Since this fact does not seem to have been explicitly proved before, we sketch the proof. We know that $\mathbf{r}$ is 2-automatic. If $\mathbf{r}$ were the fixed point of a k-uniform morphism for some k not a power of 2, then by Cobham's celebrated theorem [6], $\mathbf{r}$ would be ultimately periodic, which it is not (since its critical exponent is 4). So it must be the fixed point of a morphism h that is 2^k-uniform for some $k \geq 1$. Now $\mathbf{r}$ starts $\mathtt{00}$; if $\mathbf{r} = h(\mathbf{r})$ then $\mathbf{r}$ starts with $h(\mathtt{0})h(\mathtt{0})$. This means $\mathbf{r}[2^k - 1] = \mathbf{r}[2^{k+1} - 1]$. But clearly the number of occurrences of $\mathtt{11}$ in $2^k - 1$ is one less than the number of occurrence of $\mathtt{11}$ in $2^{k+1} - 1$, a contradiction.)

Allouche, Rampersad, and Shallit [2] proved that the question

Given a rational number $r > 1$, is $\mathbf{a}$ r-power-free?

is recursively solvable for k-automatic sequences $\mathbf{a}$. More recently, Charlier, Rampersad, and Shallit [5] showed that

Given $\mathbf{a}$, is its critical exponent infinite?

also has a recursive solution for k-automatic sequences.

In this paper we show, generalizing some of the results of Krieger mentioned above, that the critical exponent of a k-automatic sequence is always either rational or infinite. Furthermore, we show that the question

Given **a**, what is its critical exponent?

is recursively solvable for k-automatic sequences.

2 Two-dimensional automata

In this paper, we always assume that numbers are encoded in base k using the digits in $\Sigma_k = \{0,1,\ldots,k-1\}$.

The *canonical encoding* of n is the one with no leading zeroes, and is denoted $(n)_k$. Thus, for example, we have $(43)_2 = \texttt{101011}$. Similarly, if w is a word over Σ_k, then $[w]_k$ denotes the integer represented by w in base k. Thus $[\texttt{101011}]_2 = 43$.

We will need to encode pairs of integers. We handle these by first padding the representation of the smaller integer with leading zeroes, so it has the same length as the larger one, and then coding the pair as a word over Σ_k^2. This gives the *canonical encoding* of a pair (m,n), and is denoted $(m,n)_k$. Note that the canonical encoding of a pair does not begin with a symbol that has 0 in both components. For example, the canonical representation of the pair $(20,13)$ in base 2 is

$$[\texttt{1},\texttt{0}][\texttt{0},\texttt{1}][\texttt{1},\texttt{1}][\texttt{0},\texttt{0}][\texttt{0},\texttt{1}],$$

where the first components spell out $\texttt{10100}$ and the second components spell out $\texttt{01101}$. Sometimes, by abusing notation, we will write this as $(\texttt{10100},\texttt{01101})$.

Given a finite word $x \in (\Sigma_k^2)^*$, with second component representing a number other than 0, we define $f(x) = m/n$, where m is the integer represented by the first component of x and n is the integer represented by the second component of x. Without further comment we will always assume that the words we discuss have a second component representing a nonzero number.

Usually we will assume that the base-k representation is given with the most significant digit first, but sometimes, as in the following result, it is easier to deal with the reversed representations, where the least significant digit appears first (and shorter representations, if necessary, are padded with trailing zeroes). Since the class of regular languages is (effectively) closed under the map $L \to L^R$ that sends a regular language to its reversal, this distinction is not crucial to our results, and we will not emphasize it unduly.

Lemma 7. *Let β be a non-negative real number and define*

$$L_{\leq\beta} = \{x \in (\Sigma_k^2)^* \ : \ f(x) \leq \beta\},$$

and analogously for the other relations such as $<,=,\geq,>$. Then $L_{\leq\beta}$ (resp., $L_{<\beta}$, $L_{=\beta}$, $L_{\geq\beta}$, $L_{>\beta}$) is regular iff β is a rational number.

Proof. We handle only the case $L_{\leq\beta}$, the others being similar.

Suppose β is rational. Then we can write $\beta = P/Q$ for integers $P \geq 0$, $Q \geq 1$. On input x representing a pair of integers (p,q) in the reversed base-k representation, we need to accept iff $p/q \leq P/Q$, that is, iff $pQ \leq qP$. To do so, we simply transduce p and q to pQ and qP on the fly, respectively, and compare them digit-by-digit. Minor complications arise if the base-k expansions of pQ and qP have different numbers of digits. To handle this, we accept if some path ending in $[0,0]^i$ leads to an accepting condition. This construction was already given in [2], and the details can be found there.

For the other direction, we use ordinary (most-significant-digit first) representation. Without loss of generality, we can assume $1/k \leq \beta < 1$; if not, we can ensure this condition holds by modifying the automaton, shifting one coordinate to the left or right. Take $L_{\leq\beta}$ and intersect with the (regular) language of words whose second coordinates are of the form 10^*; then project onto the first coordinate to get L', a regular language over Σ_k. Now take the lexicographically largest word of each length in L' to get L''; by a well-known result (e.g., [16]), this language is also regular. But L'' has exactly one word of each length, so by another well-known result (e.g., [13, 15, 16]), L'' must be a finite union of languages of the form uv^*w. But then β is rational, as it is given by a number whose base-k representation is $.uvvv\cdots$ for some u,v. $\square$

$\square$

Theorem 8. *Let $L \subseteq (\Sigma_k^2)^*$ be a regular language. Then $\sup_{x\in L} f(x)$ is either infinite or rational.*

Proof. We assume — in order to derive a contradiction — that

$$\alpha := \sup_{x\in L} f(x)$$

is finite and irrational. Let L be accepted by a DFA M of n states.

The basic idea of the proof is simple: if α is irrational, then we can find an element $x \in L$ with $f(x)$ arbitrarily close to α. Using an argument like the classical pumping lemma for regular languages, we show how to find a new x' with $f(x') > \alpha$, a contradiction.

Define

$$\delta = \min_{\substack{0\leq r<k^n \\ 1\leq s<k^n}} \left|\alpha - \frac{r}{s}\right|.$$

Clearly $\delta > 0$ since α is irrational.

Choose $x = (p,q)_k \in L$ such that

$$0 < \alpha - \frac{p}{q} < k^{-(n+2)}\delta. \tag{1}$$

Without loss of generality, we may also assume that $|x| \geq n$ and if $y \in L$ and $|y| < |x|$ then $f(y) < f(x)$.

Now consider the path labeled x in M. By the pumping lemma, since $|x| \geq n$, we can write $x = uvw$ for $|uv| \leq n$ such that $uv^iw \in L$ for all $i \geq 0$. We now claim

Lemma 9. *Exactly one of the following cases holds:*
(a) We have $f(uv^iw) = f(uv^{i+1}w)$ for all $i \geq 0$;
(b) We have $f(uv^iw) < f(uv^{i+1}w)$ for all $i \geq 0$;
(c) We have $f(uv^iw) > f(uv^{i+1}w)$ for all $i \geq 0$.

Proof. Let us extract the first and second coordinates from the words u, v, and w as follows: $u = (u_1,u_2)$, $v = (v_1,v_2)$, and $w = (w_1,w_2)$. Then

$$\begin{aligned} f(uv^{i+1}w) &= \frac{[u_1v_1^{i+1}w_1]_k}{[u_2v_2^{i+1}w_2]_k} \\ &= \frac{([u_1v_1]_k - [u_1]_k)\cdot k^{i|v_1|+|w_1|} + [u_1v_1^iw_1]_k}{([u_2v_2]_k - [u_2]_k)\cdot k^{i|v_2|+|w_2|} + [u_2v_2^iw_2]_k}. \end{aligned}$$

It follows that

$$f(uv^{i+1}w) - f(uv^i w) = \frac{B(AE - CD)}{E(DB+E)}, \tag{2}$$

where $a = |v_1| = |v_2|$ and $b = |w_1| = |w_2|$ and

$$\begin{aligned} A &= [u_1 v_1]_k - [u_1]_k \\ B &= k^{ia+b} \\ C &= [u_1 v_1^i w_1]_k \\ D &= [u_2 v_2]_k - [u_2]_k \\ E &= [u_2 v_2^i w_2]_k. \end{aligned}$$

If $[u_2]_k$ and $[v_2]_k$ are both 0, then the second component of x begins with 0. Since we are using the canonical representation, this means the first component begins with a nonzero digit (for otherwise the representation would begin with leading zeroes). Hence $f(uv^i w) \to \infty$ as $i \to \infty$, contradicting our hypothesis. Therefore $[u_2]_k(k^a - 1) + [v_2]_k$ is positive, and hence $D > 0$. Thus the sign of $f(uv^{i+1}w) - f(uv^i w)$ is the same as the sign of $AE - CD$, so it suffices to show this quantity is independent of i.

Writing $U_1 = [u_1]_k$, $V_1 = [v_1]_k$, $W_1 = [w_1]_k$, and similarly for U_2, V_2, W_2, we find

$$\begin{aligned} A &= U_1 \cdot k^a + V_1 - U_1 \\ C &= U_1 \cdot k^{ia+b} + V_1 \cdot k^b \cdot \frac{k^{ia} - 1}{k^a - 1} + W_1 \\ D &= U_2 \cdot k^a + V_2 - U_2 \\ E &= U_2 \cdot k^{ia+b} + V_2 \cdot k^b \cdot \frac{k^{ia} - 1}{k^a - 1} + W_2. \end{aligned}$$

A tedious calculation gives

$$\begin{aligned} AE - CD = (k^a - 1)^{-1} \Big(& (k^{2a}(U_1 W_2 - U_2 W_1) + k^{a+b}(U_2 V_1 - U_1 V_2) \\ & + k^a(2U_2 W_1 - V_2 W_1 - 2U_1 W_2 + V_1 W_2) + k^b(U_1 V_2 - U_2 V_1) \\ & + U_1 W_2 + V_2 W_1 - V_1 W_2 - U_2 W_1) \Big), \end{aligned}$$

which is indeed independent of i. □

□

Now if case (a) or (c) of Lemma 9 applies, then uw is a shorter word with $f(uw) \geq p/q$, contradicting our assumption that $x = uvw$ was the shortest such.

Otherwise case (b) applies. Suppose

$$\left| \frac{p}{q} - \frac{A}{D} \right| \leq \frac{\delta}{2}. \tag{3}$$

From (1) we have $|\alpha - \frac{p}{q}| < k^{-(n+2)}\delta < \frac{\delta}{2}$ and hence, combining with (3) and using the triangle inequality, we get

$$\left| \alpha - \frac{A}{D} \right| < \delta$$

with $|A|, |D| < k^n$. But this contradicts the definition of δ.

Hence

$$\left|\frac{p}{q} - \frac{A}{D}\right| > \frac{\delta}{2}.$$

From the inequality of case (b) we get $\frac{p}{q} = f(uvw) < f(uv^2w)$. Now

$$q \le k^{|x|} = k^{|uvw|} = k^{|vw|+|u|} \le k^{a+b+n}. \tag{4}$$

Furthermore, by (2) with $i = 1$ we have

$$\begin{aligned}
f(uv^2w) - f(uvw) &= \frac{B(AE-CD)}{E(DB+E)} \\
&= \frac{k^{a+b}(Aq-pD)}{q(D\cdot k^{a+b}+q)} \\
&> \frac{\frac{\delta}{2}Dqk^{a+b}}{q(q+k^{a+b}D)} \\
&= \frac{\frac{\delta}{2}Dk^{a+b}}{q+k^{a+b}D} \\
&\ge \frac{\frac{\delta}{2}Dk^{a+b}}{k^{a+b+n}+k^{a+b}D} \\
&= \frac{\frac{\delta}{2}}{\frac{k^n}{D}+1} \\
&\ge \frac{\frac{\delta}{2}}{k^n+1} \\
&\ge \frac{\frac{\delta}{2}}{2k^n} \\
&\ge k^{-(n+2)}\delta,
\end{aligned}$$

so $f(uv^2w) > \frac{p}{q} + k^{-(n+2)}\delta > \alpha$, a contradiction.

This completes the proof. □

□

Next, we prove a decidability result.

Theorem 10. *Given an automaton M accepting L, the quantity $\alpha = \sup_{(p,q)_k \in L} p/q$ is computable.*

Proof. It suffices to produce a finite set S of explicitly-computable rational numbers in which α must lie. For once we do this, using Lemma 7, we can intersect L with $L_{>\beta}$ for each $\beta \in S$; then α equals the smallest β for which this intersection is empty.

Suppose M has n states. We claim that we can take

$$S = S_1 \cup S_2$$

where

$$\begin{aligned} S_1 &= \{p/q \;:\; 0 \leq p < k^n,\ 1 \leq q < k^n\} \\ S_2 &= \left\{ \frac{[u_1]_k + \frac{[v_1]_k}{k^a-1}}{[u_2]_k + \frac{[v_2]_k}{k^a-1}} \;:\; |u_1v_1| \leq n, |u_2v_2| \leq n, |u_1| = |u_2|, |v_1| = |v_2| = a \geq 1 \right\}. \end{aligned}$$

There are two cases to consider:

Case 1: $\alpha = \sup_{(p,q)_k \in L} p/q$ is achieved on some $x \in L$. Without loss of generality we can assume this is the shortest word achieving the sup.

Suppose (to get a contradiction) that $|x| \geq n$. Then as in the proof of Theorem 8, we can use the pumping lemma to write $x = uvw$ with $|uv| \leq n$ and $|v| \geq 1$. If $f(uw) = \alpha$, then we have found a shorter word achieving the sup, a contradiction. If $f(uw) > f(uvw)$, then $f(uvw) = \alpha$ is not the sup. Hence $f(uw) < f(uvw)$. Then by Lemma 9 we must have $f(uv^2w) > f(uvw) = \alpha$, a contradiction.

Since the length of the word achieving the sup is at most n, we must have $\alpha \in S_1$.

Case 2: The sup is not achieved on L. Choose a sequence $x_i \in L$ with $f(x_i)$ converging strictly monotonically to α, with $f(x_i) > f(y)$ for all $y \in L$ with $|y| < |x_i|$, and such that each x_i is of length $\geq n$.

As before, use the pumping lemma to write $x_i = uvw$ with $|uv| \leq n$ and $|v| \geq 1$. If $f(uw) \geq f(uvw) = f(x_i)$, then this contradicts our assumption about $f(x_i)$. So $f(uw) < f(uvw)$. But then, by Lemma 9 we have $f(x_i) = f(uvw) < f(uv^jw)$ for all $j \geq 2$. Now let $j \to \infty$. Then $f(uv^jw)$ converges to

$$g(u,v) := \frac{[u_1]_k + \frac{[v_1]_k}{k^a-1}}{[u_2]_k + \frac{[v_2]_k}{k^a-1}},$$

where $u = (u_1, u_2)$, $v = (v_1, v_2)$, and $a = |v_1| = |v_2|$. Each uv^jw lies in L, so $\sup_{x \in L} f(x) \geq g(u,v)$ for each of the u, v considered above. On the other hand, since the x_i converge to α, we see that α must equal the sup of $g(u,v)$ over all u, v corresponding to an x_i. Thus $\alpha \in S_2$.

This completes the proof. □

□

Corollary 11. *Let L be accepted by a DFA with n states. If $\frac{p}{q} = \sup_{x \in L} f(x)$ is attained by some $x \in L$, then $p, q < k^n$. If the sup is not attained, then $p, q < k^{2n}$.*

3 Application to the critical exponent

We can now apply the results of the previous section to the critical exponent problem.

Theorem 12. *Given a k-automatic sequence we can effectively compute its critical exponent.*

Proof. Given a k-automatic sequence $\mathbf{a} = (a_i)_{i \geq 0}$, we can, using the techniques of [2, 5], (effectively) create a two-dimensional DFA M accepting

$$L = \{(p,q)_k \;:\; \exists \text{ a factor of } \mathbf{a} \text{ of length } q \text{ with period } p\ \}.$$

To do so, on input $(p,q)_k$, we nondeterministically choose an index i at which a factor of length q begins in $\mathbf{a}$, and then verify that $\mathbf{a}[j] = \mathbf{a}[j+p]$ for $i \leq j \leq i+q-p-1$.

Then the critical exponent of $\mathbf{a}$ is $\sup_{x \in L(M)} f(x)$, which, as we have seen, is either rational or infinite. The infinite case has already been handled in [5]. In the finite case, Theorem 8 tells us that the critical exponent is rational, and Theorem 10 tells us how to compute it from M. □

□

Remark 13. The same results also hold for some variations of the critical exponent, such as when the sup is taken over only the factors that occur infinitely often. They also hold for the "initial critical exponent", where the supremum is taken over all *prefixes* of a given infinite word, as opposed to all factors.

Remark 14. We can obtain the same results for $\limsup_{x\in L} f(x)$ instead of $\sup_{x\in L} f(x)$. Here, to get a contradiction, we assume that α is the lim sup, and then we construct an infinite sequence of distinct points lying in a bounded interval $> \alpha$.

Remark 15. In [4] the authors studied $\limsup_{k\to\infty}(j_k/i_k)$, where (i_k, j_k) are the starting and ending indices of the k'th maximal block of letters chosen from some subalphabet Δ' in a morphic word. Our technique also applies here if the underlying sequence is k-automatic.

4 Other applications

The results in this paper have applications to other problems.

A sequence $\mathbf{a}$ is said to be *recurrent* if every factor that occurs, occurs infinitely often. It is *linearly recurrent* if there exists a constant C such that for all $\ell \geq 0$, and all factors x of length ℓ occurring in $\mathbf{s}$, any two consecutive occurrences of x are separated by at most $C\ell$ positions.

Theorem 16. *It is decidable if a k-automatic sequence* $\mathbf{a}$ *is linearly recurrent. If* $\mathbf{a}$ *is linearly recurrent, the optimal constant C is computable.*

Proof. First, as in [5], we construct an automaton accepting the language

$$\begin{aligned} L = \{(n,l)_k \ : \ & \text{(a) there exists } i \geq 0 \text{ s. t. for all } j, 0 \leq j < \ell \text{ we have } \mathbf{a}[i+j] = \mathbf{a}[i+n+j] \text{ and} \\ & \text{(b) there is no } t, 0 < t < n \text{ s. t. for all } j, 0 \leq j < \ell \text{ we have } \mathbf{a}[i+j] = \mathbf{a}[i+t+j]\ \} \end{aligned}$$

Another way to say this is that L consists of the base-k representation of those pairs of integers (n,ℓ) such that (a) there is some factor of length ℓ for which there is another occurrence at distance n and (b) this occurrence is actually the very next occurrence.

Now from Theorem 10 we know that $\sup\{n/\ell \ : \ (n,\ell)_k \in L\}$ is either infinite or rational. In the latter case this sup is computable, and this gives the optimal constant C for the linear recurrence of $\mathbf{a}$. □

5 Open problems

In this paper we have examined $\sup_{x\in L} f(x)$. We do not know how to extend these results to the more general case of morphic sequences.

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